REPORT No. 794

THE FLOW OF A COMPRESSIBLE FLUID PAST A CIRCULAR ARC PROFILE

By Carl Kaplan

SUMMARY

The Ackeret iteration process is utilized to obtain higher approximations than that of Prandtl and Glauert for the flow of a compressible fluid past a circular arc profile. The procedure is to expand the velocity potential in a power series of the camber coefficient. The first two terms of the development correspond to the Prandtl-Glauert approximation and yield the well-known correction to the circulation about the profile. The second approximation, involving the square of the camber coefficient, improves the velocity and pressure fields but yields no new results with regard to the circulation, since the circulation about the profile is an odd function of the camber coefficient. The third approximation, involving the cube of the camber coefficient, permits the use of higher values of the camber coefficient and furthermore yields an improvement to the Prandtl-Glauert rule with regard to the effect of compressibility on the circulation of the circular arc profile. Numerical examples with tables and graphs illustrate the results of the analysis.

INTRODUCTION

The calculation of the two-dimensional steady flow of a compressible fluid past a prescribed body can be performed by a method independently discovered by Janzen (reference 1) and by Rayleigh (reference 2), which consists in developing the velocity potential or the stream function according to powers of the stream Mach number. The first approximation is the incompressible case and the succeeding approximations represent the effect of compressibility. The method has in recent years been successively improved by Poggi (reference 3), by Imai and Aihara (reference 4), and by the present author (reference 5). Although the method can be applied to an arbitrary profile, it suffers from the practical restriction to small stream Mach numbers, because approximations beyond the second or third entail a prohibitive amount of labor.

For the flow past a profile of small thickness, camber, and angle of attack, Prandtl (reference 6), Glauert (reference 7), and Ackeret (reference 8) obtained by various means an approximation that applies to the entire subsonic range of velocity. The present author (reference 9) extended the method of Ackeret by an iteration process that takes into account the effect of thickness and applied the method to a particular family of symmetrical profiles. In the present paper, the effect of camber is investigated by a similar application of the method of reference 9 to a circular arc profile. In the application of the method, it is desirable to avoid stagnation points so that the variation of the local velocity from that of the undisturbed stream can be made small. For this reason the direction of the undisturbed stream is chosen parallel to the chord of the circular arc (ideal angle of attack) and the circulation about the profile is determined in accordance with the Kutta condition; namely, that the flow past the profile leave the trailing edge tangentially. The flow is symmetrical fore and aft and the velocity remains finite at all points. The circulation in a compressible flow will be seen to be an odd function of the camber coefficient. In order, then, to obtain an improvement of the Prandtl-Glauert rule, it is necessary to carry the iteration process through three approximations.

THE ITERATION PROCESS

The velocity potential ϕ (X, Y) of the two-dimensional, steady, irrotational flow of a compressible fluid satisfies the following differential equation of the second order:

$$(c^2-u^2)\frac{\partial^2\phi}{\partial X^2}-2uv\frac{\partial^2\phi}{\partial X\,\partial Y}+(c^2-v^2)\frac{\partial^2\phi}{\partial Y^2}=0 \tag{1}$$

where

X, Y rectangular Cartesian coordinates in plane of flow

$u=\frac{\partial\phi}{\partial X}$, $v=\frac{\partial\phi}{\partial Y}$ fluid velocity components along X- and Y-axes, respectively

c local velocity of sound

The local velocity of sound c is expressed in terms of the fluid velocity q by means of Bernoulli's equation

$$\int_{p_1}^{p}\frac{dp}{\rho}+\frac{1}{2}q^2=0 \tag{2}$$

the equation defining the velocity of sound

$$c^2=\frac{dp}{d\rho} \tag{3}$$

and the adiabatic relation between the pressure and the density

$$\frac{p}{p_1}=\left(\frac{\rho}{\rho_1}\right)^\gamma \tag{4}$$

In equations (2), (3), and (4),

p static pressure in fluid
p_1 static pressure in undisturbed stream at infinity
ρ density of fluid
ρ_1 density of undisturbed stream at infinity
q magnitude of velocity of fluid
γ adiabatic index (approx. 1.4 for air)

For the adiabatic case, equation (3) yields

$$c^2=\gamma\frac{p}{\rho} \tag{5}$$

By means of equations (4) and (5) Bernoulli's equation, equation (2), yields the following relations:

$$\left.\begin{aligned}c^2&=c_1^2\left[1-\frac{\gamma-1}{2}M_1^2\left(\frac{q^2}{U^2}-1\right)\right]\\ \rho&=\rho_1\left[1-\frac{\gamma-1}{2}M_1^2\left(\frac{q^2}{U^2}-1\right)\right]^{\frac{1}{\gamma-1}}\\ p&=p_1\left[1-\frac{\gamma-1}{2}M_1^2\left(\frac{q^2}{U^2}-1\right)\right]^{\frac{\gamma}{\gamma-1}}\end{aligned}\right\}\tag{6}$$

where

U velocity of undisturbed stream at infinity
c_1 velocity of sound in undisturbed stream at infinity
M_1 Mach number of undisturbed stream at infinity

Now, if the profile is held fixed in the uniform stream of velocity U and if a characteristic length s is assumed to be the unit of length and the stream velocity U is assumed to be the unit of velocity, the fundamental differential equation (1) and the first of equations (6) become

$$\left(\frac{c^2}{c_1^2}-M_1^2u^2\right)\frac{\partial^2\phi}{\partial X^2}-2M_1^2uv\frac{\partial^2\phi}{\partial X\,\partial Y}+\left(\frac{c^2}{c_1^2}-M_1^2v^2\right)\frac{\partial^2\phi}{\partial Y^2}=0\tag{7}$$

and

$$\frac{c^2}{c_1^2}=1-\frac{\gamma-1}{2}M_1^2(q^2-1)\tag{8}$$

where X, Y, u, v, q, and ϕ now denote, respectively, the nondimensional quantities X/s, Y/s, u/U, v/U, q/U, and ϕ/Us.

The iteration process consists in developing the velocity potential ϕ in powers of a parameter h, the camber of the circular arc profile. Thus

$$\phi=-X-h\phi_1-h^2\phi_2-h^3\phi_3-\ \cdot\ \cdot\ \cdot\tag{9}$$

and

$$\left.\begin{aligned}u&=-1-h\frac{\partial\phi_1}{\partial X}-h^2\frac{\partial\phi_2}{\partial X}-h^3\frac{\partial\phi_3}{\partial X}-\ \cdot\ \cdot\ \cdot\\ v&=-h\frac{\partial\phi_1}{\partial Y}-h^2\frac{\partial\phi_2}{\partial Y}-h^3\frac{\partial\phi_3}{\partial Y}-\ \cdot\ \cdot\ \cdot\end{aligned}\right\}\tag{10}$$

When these expressions for ϕ, u, and v, together with the expression for c^2/c_1^2 given by equation (8), are introduced into the fundamental differential equation (7) and when the coefficients of the various powers of h are equated to zero, the following differential equations for ϕ_1, ϕ_2, ϕ_3, . . . result:

$$(1-M_1^2)\frac{\partial^2\phi_1}{\partial X^2}+\frac{\partial^2\phi_1}{\partial Y^2}=0\tag{11}$$

$$(1-M_1^2)\frac{\partial^2\phi_2}{\partial X^2}+\frac{\partial^2\phi_2}{\partial Y^2}=M_1^2\left[(\gamma+1)\frac{\partial\phi_1}{\partial X}\frac{\partial^2\phi_1}{\partial X^2}+(\gamma-1)\frac{\partial\phi_1}{\partial X}\frac{\partial^2\phi_1}{\partial Y^2}+2\frac{\partial\phi_1}{\partial Y}\frac{\partial^2\phi_1}{\partial X\,\partial Y}\right]\tag{12}$$

$$\begin{aligned}(1-M_1^2)\frac{\partial^2\phi_3}{\partial X^2}+\frac{\partial^2\phi_3}{\partial Y^2}=M_1^2\Bigg\{&\frac{1}{2}\left(\frac{\partial\phi_1}{\partial X}\right)^2\left[(\gamma+1)\frac{\partial^2\phi_1}{\partial X^2}+(\gamma-1)\frac{\partial^2\phi_1}{\partial Y^2}\right]+\frac{1}{2}\left(\frac{\partial\phi_1}{\partial Y}\right)^2\left[(\gamma-1)\frac{\partial^2\phi_1}{\partial X^2}+(\gamma+1)\frac{\partial^2\phi_1}{\partial Y^2}\right]\\ &+\frac{\partial\phi_2}{\partial X}\left[(\gamma+1)\frac{\partial^2\phi_1}{\partial X^2}+(\gamma-1)\frac{\partial^2\phi_1}{\partial Y^2}\right]+\frac{\partial\phi_1}{\partial X}\left[(\gamma+1)\frac{\partial^2\phi_2}{\partial X^2}+(\gamma-1)\frac{\partial^2\phi_2}{\partial Y^2}\right]+2\left(\frac{\partial\phi_1}{\partial X}\frac{\partial\phi_1}{\partial Y}\frac{\partial^2\phi_1}{\partial X\,\partial Y}+\frac{\partial\phi_2}{\partial Y}\frac{\partial^2\phi_1}{\partial X\,\partial Y}+\frac{\partial\phi_1}{\partial Y}\frac{\partial^2\phi_2}{\partial X\,\partial Y}\right)\Bigg\}\end{aligned}\tag{13}$$

These differential equations may be put into more familiar forms by the introduction of a new set of independent variables x and y, where

$$\left.\begin{aligned}x&=X\\ y&=\beta Y\end{aligned}\right\}\tag{14}$$

and

$$\beta=(1-M_1^2)^{1/2}$$

For $M_1<1$, equation (11) then becomes a Laplace equation and equations (12) and (13) become Poisson equations. Equation (11) replaces the fundamental differential equation (7) for flows that differ only slightly from the undisturbed stream, and its solution yields the well-known Prandtl-Glauert result. The solutions of equations (12) and (13) provide successive improvements in the approximation to the solution of a compressible-flow problem.

For the present problem, the procedure to be followed in solving equations (11) to (13) is first to obtain the velocity potential for the incompressible case in the form of a power series in the camber coefficient h of the circular arc profile. The solution for the first approximation ϕ_1 of the compressible flow is then obtained by analogy from the form of the coefficient of h of the incompressible velocity potential. The solutions of equations (12) and (13) for the second and third approximations ϕ_2 and ϕ_3 follow by a straight-forward procedure. The boundary conditions—that the flow be tangential to the profile and that the disturbance to the main stream vanish at infinity—are satisfied to the same power of the camber coefficient h that is involved in the approximation for the velocity potential ϕ. The calculations are laborious when more than two steps in the iteration process are involved but the third step is necessary to obtain results that extend present-day knowledge. Most of the details of calculation are given in appendixes in order not to obscure the presentation of the main results.

RESULTS OF THE ANALYSIS

Expression for the velocity potential.—The choice of the circular arc as the solid boundary was made for two reasons: (1) The solution of the incompressible flow can be easily expressed in a closed form, and (2) when the circular arc is fixed in a uniform stream flowing in a direction parallel to the chord and when the Kutta condition—that the flow leave the trailing edge tangentially—is applied, the velocities at the nose and the tail are finite and different from zero. No stagnation points occur, therefore, on the boundary or in the field of flow and a greater degree of accuracy in the iteration process is assured. Appendix A contains the calculation of the incompressible flow past the circular arc profile and appendixes B, C, and D contain the detailed calculations for ϕ_1, ϕ_2, and ϕ_3, respectively. The final expression for the velocity potential ϕ takes the following form:

$$\phi=-\cosh\xi\cos\eta-h\phi_1-h^2\phi_2-h^3\phi_3-\ \cdot\ \cdot\ \cdot\tag{15}$$

where, from equation (B9),

$$\phi_1=\frac{1}{\beta}\,(e^{-2\xi}\sin 2\eta-2\eta)$$

from equation (C13),

$$\begin{aligned}\phi_2=&(2D[(\gamma+1)D+4]\xi e^{-\xi}+2De^{-3\xi}\\&+2\{3-D+D[(\gamma+1)D+4]\}e^{-\xi})\cos\eta\\&-\left(\frac{1}{2}(\gamma+1)D^2e^{-\xi}+\frac{1}{6}\{12+12D-D[(\gamma+1)D+4]\}e^{-3\xi}\right)\cos 3\eta\end{aligned}$$

and from equation (D18),

$$\begin{aligned}\phi_3=&G_1(\xi)\sin 2\eta+G_2(\xi)\sin 4\eta\\&+G(\xi)\left(\frac{1}{2}\frac{\sin 2\eta}{\cosh 2\xi-\cos 2\eta}-e^{-2\xi}\sin 2\eta-e^{-4\xi}\sin 4\eta\right)\\&-[2G_1(0)+4G_2(0)]\eta\end{aligned}$$

In these equations

$D=\frac{1-\beta^2}{\beta^2}$

$G_1(\xi)$, $G_2(\xi)$, and $G(\xi)$ functions of ξ given by equations (D12), (D13), and (D19), respectively

ξ, η elliptic coordinates related to rectangular Cartesian coordinates X, Y by equations of transformation:

$$x=X=\cosh\xi\cos\eta$$

$$y=\beta Y=\sinh\xi\sin\eta$$

The circulation correction formula.—Equation (15) represents the solution of the fundamental differential equation (1) that satisfies the boundary conditions at the surface of the circular arc profile and at infinity insofar as the terms inclusive of the third power of the camber coefficient h are concerned. Each of the expressions ϕ_1, ϕ_2, and ϕ_3 are obtained in closed form and are finite for all values of the stream Mach number M_1 from zero up to but not including unity. The Kutta condition, which determines the circulation uniquely by stipulating a finite velocity at the sharp trailing edge of the circular arc, yields the following circulation correction formula (see equation (D36)):

$$\begin{aligned}\frac{\Gamma_c}{\Gamma_i}=\frac{1}{\beta}+\Big[&\frac{10}{3}\frac{1-\beta^2}{\beta^3}+\frac{1}{3}(\gamma+1)\frac{(1-\beta^2)^2}{\beta^5}(8+5\beta^2)\\&+\frac{1}{24}(\gamma+1)^2\frac{(1-\beta^2)^3}{\beta^7}(31+\beta^2)\Big]h^2\end{aligned}\qquad(16)$$

where Γ_c and Γ_i are, respectively, the circulations in the compressible and incompressible flows. The incompressible circulation Γ_i is proportional to the first power of h so that the compressible circulation Γ_c is an odd function of h. The second approximation of Γ_c is therefore identical with the first approximation and no departure from the Prandtl-Glauert rule is obtained until the third power of h is included. This result explains why the simple Prandtl-Glauert rule for the effect of compressibility on the circulation or lift of an airfoil has been very satisfactory.

For comparison, a formula analogous to equation (16) has been obtained by applying the von Kármán-Tsien velocity correction formula to the circular arc profile. From reference 10

$$\frac{q_c}{q_i}=\frac{1-\mu}{1-\mu q_i^2}$$

where

q_c velocity of compressible fluid

q_i velocity of incompressible fluid

$\mu=\left[\frac{M_1}{1+(1-M_1^2)^{1/2}}\right]^2$

By an elementary integration around the circle, corresponding conformally to the circular arc, the following relation is then obtained:

$$\frac{\Gamma_c}{\Gamma_i}=\frac{1-\mu}{4\mu^{1/2}\sin^2\delta}\left\{\frac{1-\mu^{1/2}\cos^2\delta}{[1-2\mu^{1/2}(1+\sin^2\delta)+\mu\cos^4\delta]^{1/2}}-\frac{1+\mu^{1/2}\cos^2\delta}{[1+2\mu^{1/2}(1+\sin^2\delta)+\mu\cos^4\delta]^{1/2}}\right\}\qquad(17)$$

where the angle δ (see fig. 1) is related to the camber coefficient h by means of the equation

$$\tan\delta=2h$$

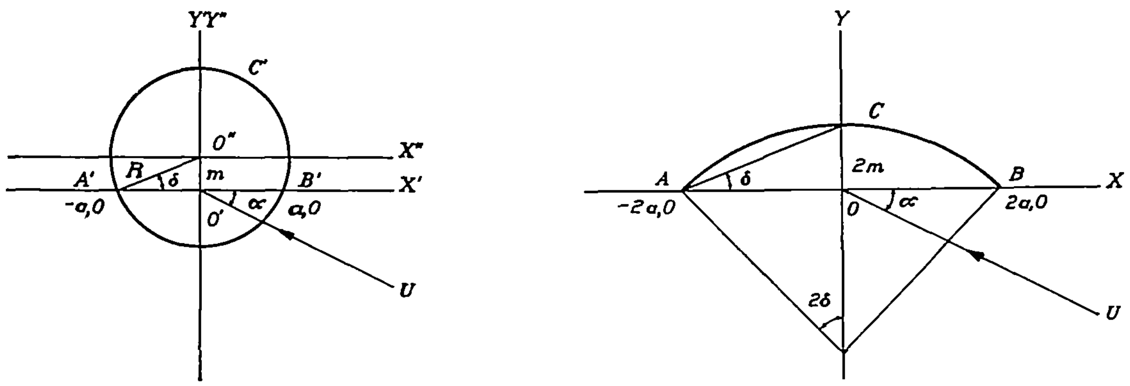

FIGURE 1.—Mapping of circular arc into circle.

Table I gives values of the ratio Γ_c/Γ_i for various values of the stream Mach number and the camber coefficient h, calculated by means of equations (16) and (17). Figure 2 shows the graphs of Γ_c/Γ_i as functions of M_1 for various values of h. The curves based on the von Kármán-Tsien velocity correction formula lie between the Prandtl-Glauert curve and the curves based on the present analysis. Figure 3 shows the graphs of Γ_c/Γ_i as functions of h for various values of M_1.

The velocity correction formula.—The magnitude of the velocity of the fluid at the surface of the circular arc is given by (see equation (D38)):

$$q=1+\frac{4h}{\beta}\sin\vartheta+h^2\left\{-2-\frac{2}{\beta^4}-(\gamma-1)\left(\frac{1-\beta^2}{\beta^2}\right)^2+4\left[\frac{2}{\beta^4}+(\gamma-1)\left(\frac{1-\beta^2}{\beta^2}\right)^2\right]\sin^2\vartheta\right\}$$

$$+h^3\left\{4\left[-\frac{2}{\beta}+G_1(0)+2G_2(0)\right]\sin\vartheta+8\left[-\frac{2}{\beta}+2\beta(2D+3)+G_2(0)\right]\sin 3\vartheta\right\}+\ .\ .\ . \tag{18}$$

where $\cos\vartheta=x$ and $0\leqq\vartheta\leqq\pi$ for the upper side of the arc, $-\pi\leqq\vartheta\leqq 0$ for the lower side of the arc, and

$$G_1(0)=-\frac{4}{\beta}+12\beta+\frac{4}{3}D\left(\frac{3}{\beta}+10\beta\right)+\frac{10}{3}\beta D^2+\frac{1}{3}D^2(\gamma+1)\left(\frac{3}{\beta}+19\beta+17\beta D\right)+\frac{1}{24}\beta D^3(\gamma+1)^2(48+47D)$$

$$G_2(0)=-\frac{1}{\beta}-3\beta-2D\left(\frac{1}{\beta}+\beta\right)-\frac{1}{2}D^2(\gamma+1)\left(\frac{1}{\beta}+2\beta+3\beta D\right)-\frac{1}{3}\beta D^3(\gamma+1)^2(1+D)$$

If q_c and q_i denote, respectively, the magnitude of the velocity at the surface of the profile in the compressible and the incompressible cases, the velocity correction formula is

$$\frac{q_c}{q_i}=\frac{q}{1+4h\sin\vartheta-4h^2\cos 2\vartheta-8h^3\sin\vartheta} \tag{19}$$

where q is obtained from equation (18). At the leading or trailing edge, $\vartheta=0$ or $\vartheta=\pi$,

$$\frac{q_c}{q_i}=\frac{1-h^2\left[\left(\frac{1}{\beta^2}+1\right)^2+\gamma\left(\frac{1}{\beta^2}-1\right)^2\right]}{1-4h^2} \tag{20}$$

At the position of maximum velocity, $\vartheta=\frac{\pi}{2}$,

$$\frac{q_c}{q_i}=\frac{1+\frac{4h}{\beta}+h^2\left[-8+3\left(\frac{1}{\beta^2}+1\right)^2+3\gamma\left(\frac{1}{\beta^2}-1\right)^2\right]+4h^3[-2\beta(4D+5)+G_1(0)]}{1+4h+4h^2-8h^3} \tag{21}$$

At the position of minimum velocity, $\vartheta=-\frac{\pi}{2}$,

$$\frac{q_c}{q_i}=\frac{1-\frac{4h}{\beta}+h^2\left[-8+3\left(\frac{1}{\beta^2}+1\right)^2+3\gamma\left(\frac{1}{\beta^2}-1\right)^2\right]-4h^3[-2\beta(4D+5)+G_1(0)]}{1-4h+4h^2+8h^3} \tag{22}$$

Tables II to IV give values of the ratio q_c/q_i based on equations (20) to (22), respectively, for the first, second, and third approximations. Figure 4 shows the graphs of $\left(\frac{q_c}{q_i}\right)_{max}-1$ as functions of M_1 for the three approximations for various values of the camber coefficient h.

The critical velocity q_{cr}, defined as the value for which the velocity of the fluid equals the local velocity of sound, is obtained from the first of equations (6) by putting $q=c=q_{cr}$. Thus

$$q_{cr}=\left(\frac{1+\frac{\gamma-1}{2}M_1^2}{\frac{\gamma+1}{2}M_1^2}\right)^{1/2} \tag{23}$$

The values of q_{cr} are given in table V in the column for which the local Mach number is unity. The ratio q_{cr}/q_i is easily calculated for the various approximations. The graphs of only the third approximation of $\frac{q_{cr}}{q_i}-1$ are included in figure 4. Table VI lists the first, second, and third approximate values of the critical stream Mach number $M_{1_{cr}}$, and figure 5 shows the corresponding graphs as functions of the camber coefficient h.

The graphs of the third approximation of the maximum and minimum values of q_c, obtained from tables III and IV, are shown in figure 6 as functions of the stream Mach number M_1. The constant local Mach number lines shown in figure 6 are obtained from equation (8) by introducing the local Mach number M in place of the local velocity of sound c. Thus

$$q=\left(\frac{\frac{\gamma-1}{2}+\frac{1}{M_1^2}}{\frac{\gamma-1}{2}+\frac{1}{M^2}}\right)^{1/2} \tag{24}$$

Note that equation (24) becomes equation (23) when $M=1$. Table V contains values of q for various values of M and M_1.

A comparison of the results of reference 9 on the compressibility effect of thickness and the results of the present paper on the compressibility effect of camber is of interest. For this purpose, a symmetrical shape of reference 9 was compared with a circular arc profile with the same incompressible maximum speed at the surface. Results of this comparison for several corresponding thickness and camber coefficients are given in table VII. The dashed curves in figure 6 are associated with the various symmetrical shapes. For moderate values of camber and thickness the difference may be seen to be negligible over the entire subsonic range. This observation indicates that, at least to a very good approximation, the effect of compressibility in the subsonic range can be considered to depend explicitly only on the incompressible fluid velocity and the stream Mach number and to be independent of the shape of the profile. This result therefore substantiates the use of velocity correction formulas such as the Prandtl-Glauert, the von Kármán-Tsien, the Temple-Yarwood, and the Garrick-Kaplan (reference 11) formulas, which depend only on the incompressible fluid velocity and on the stream Mach number.

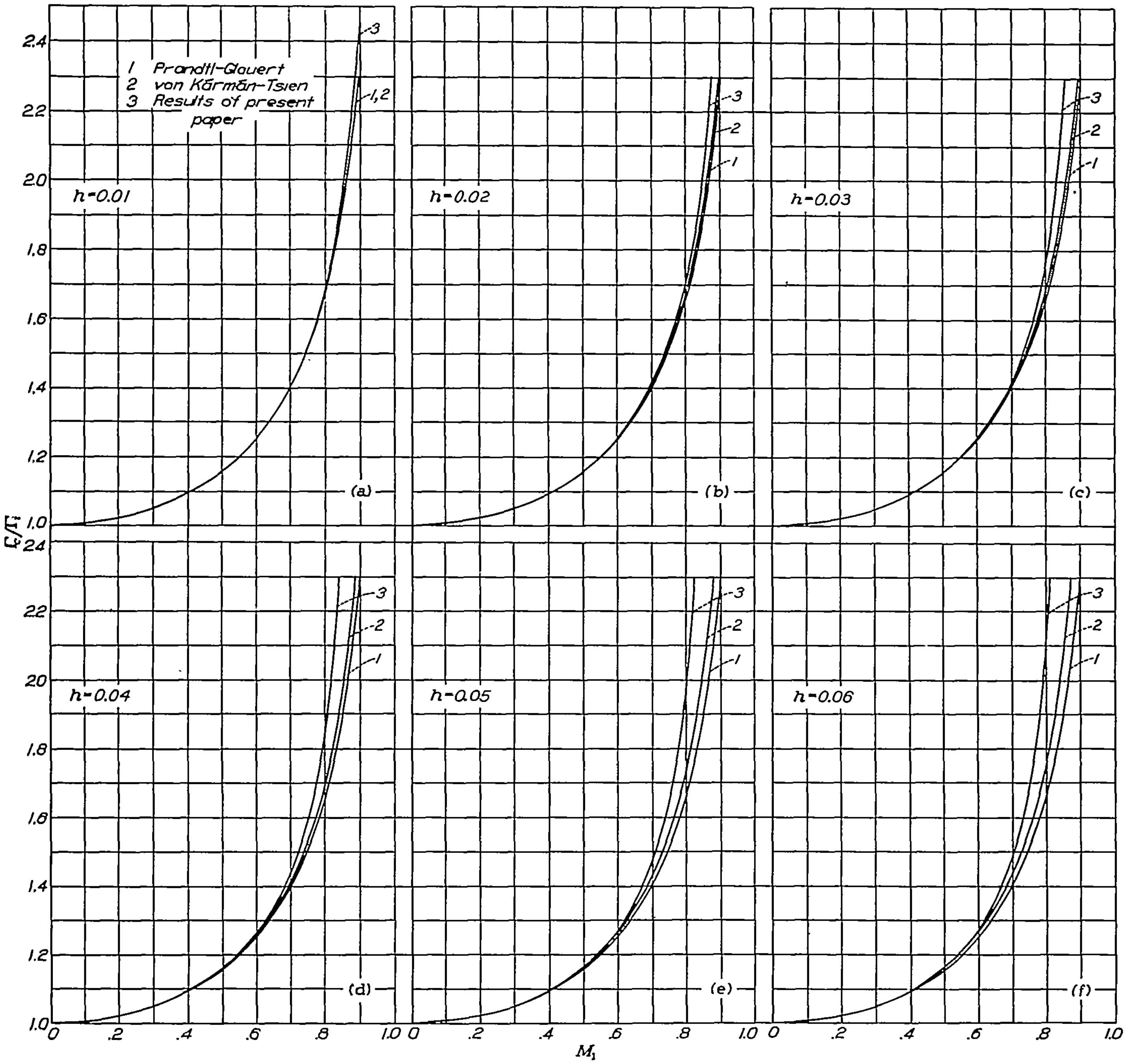

FIGURE 2.—Ratio of circulations for compressible and incompressible cases as a function of stream Mach number.

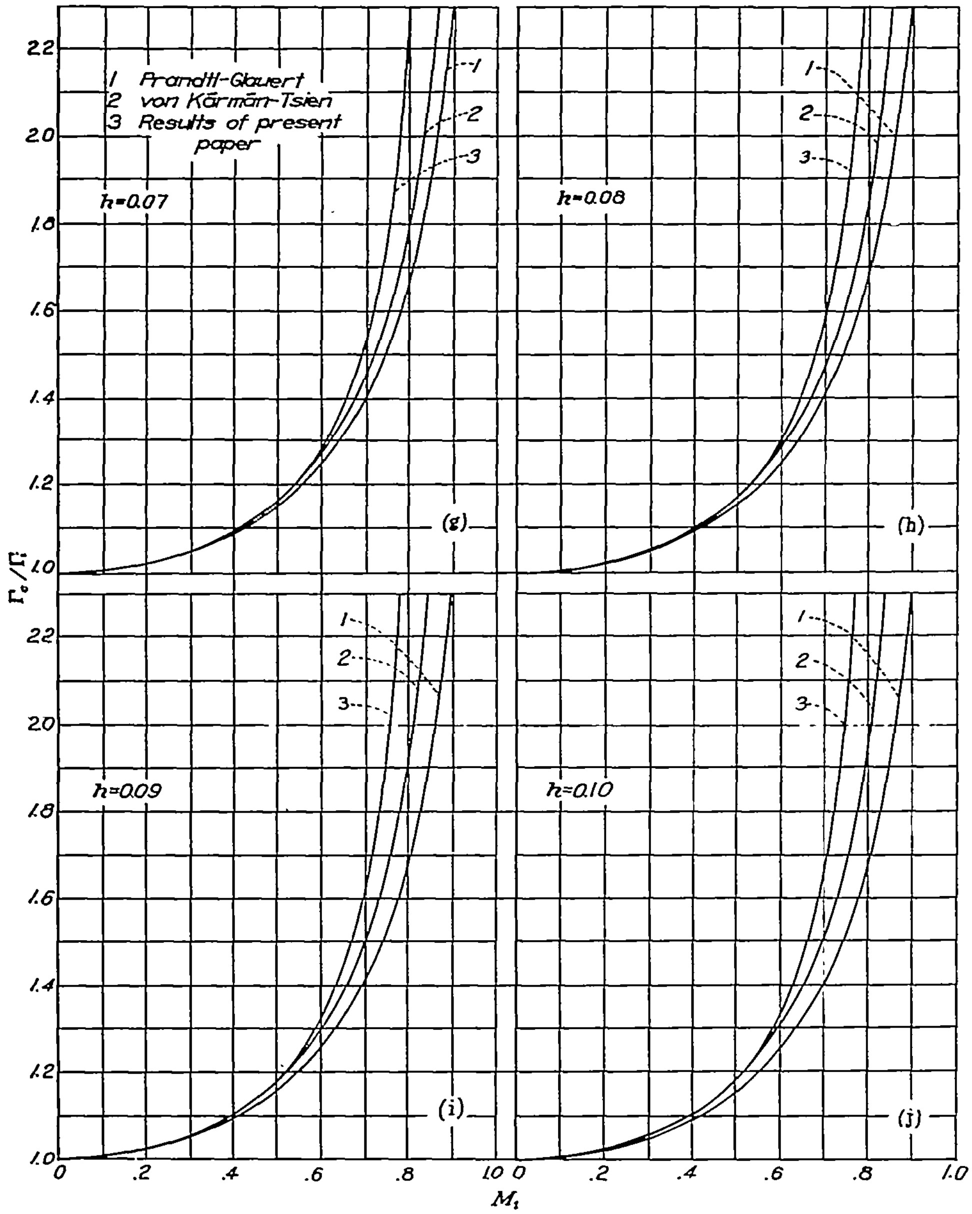

FIGURE 2.—Concluded.

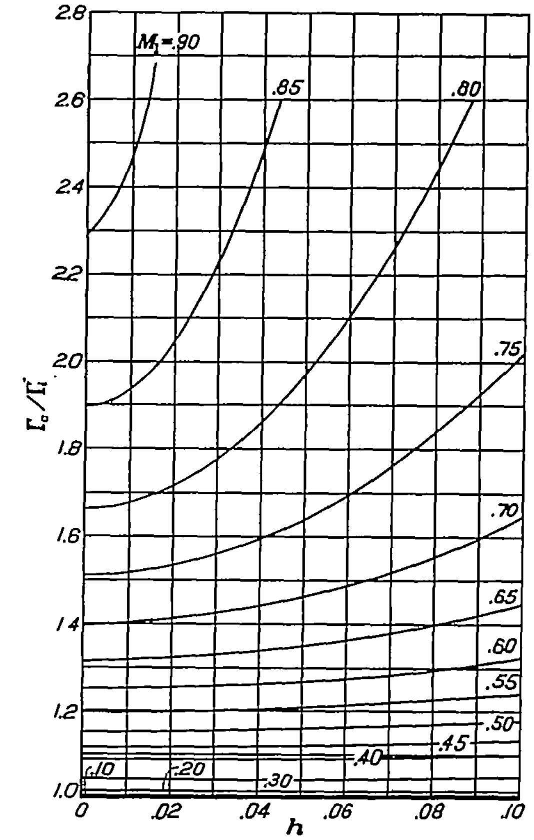

FIGURE 3.—Ratio of circulations for compressible and incompressible cases as a function of camber coefficient.

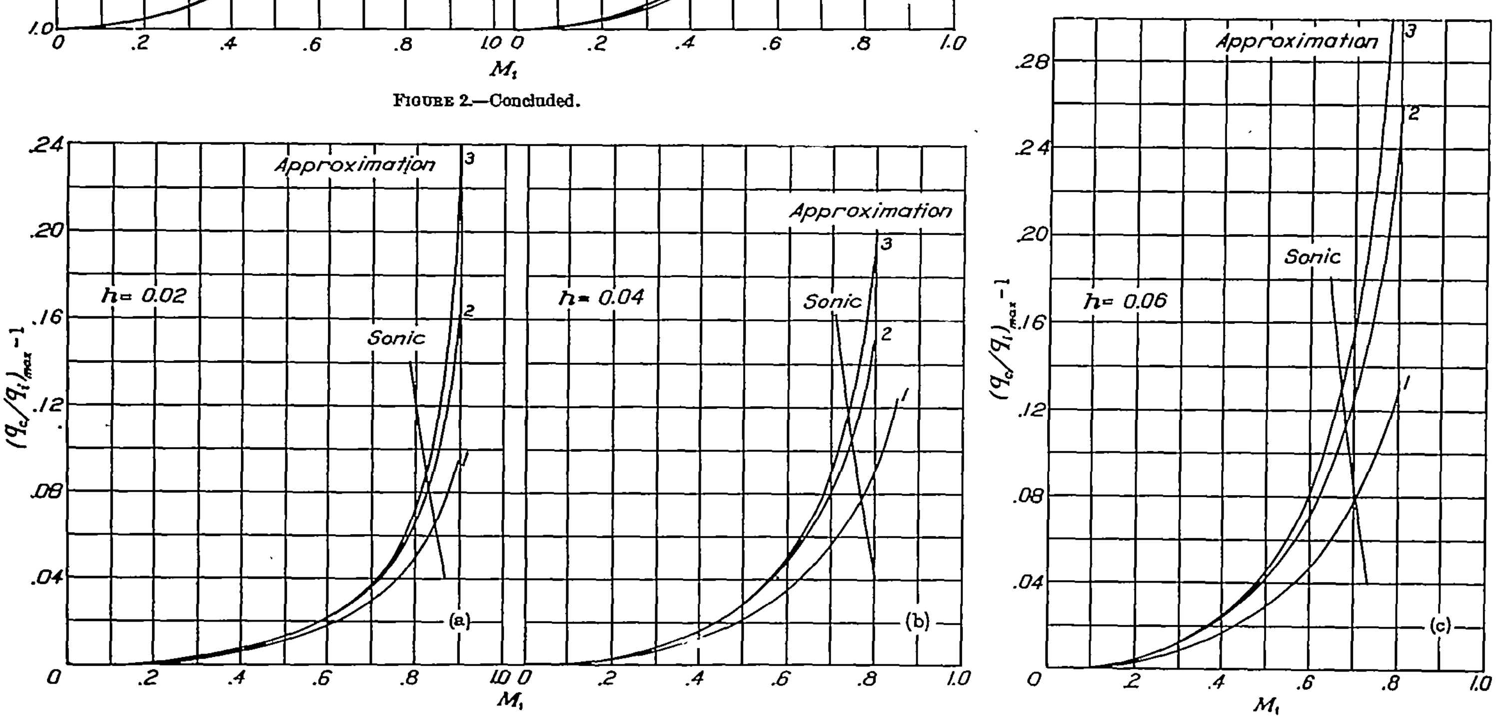

FIGURE 4.—Ratio of velocities for compressible and incompressible cases as a function of stream Mach number.

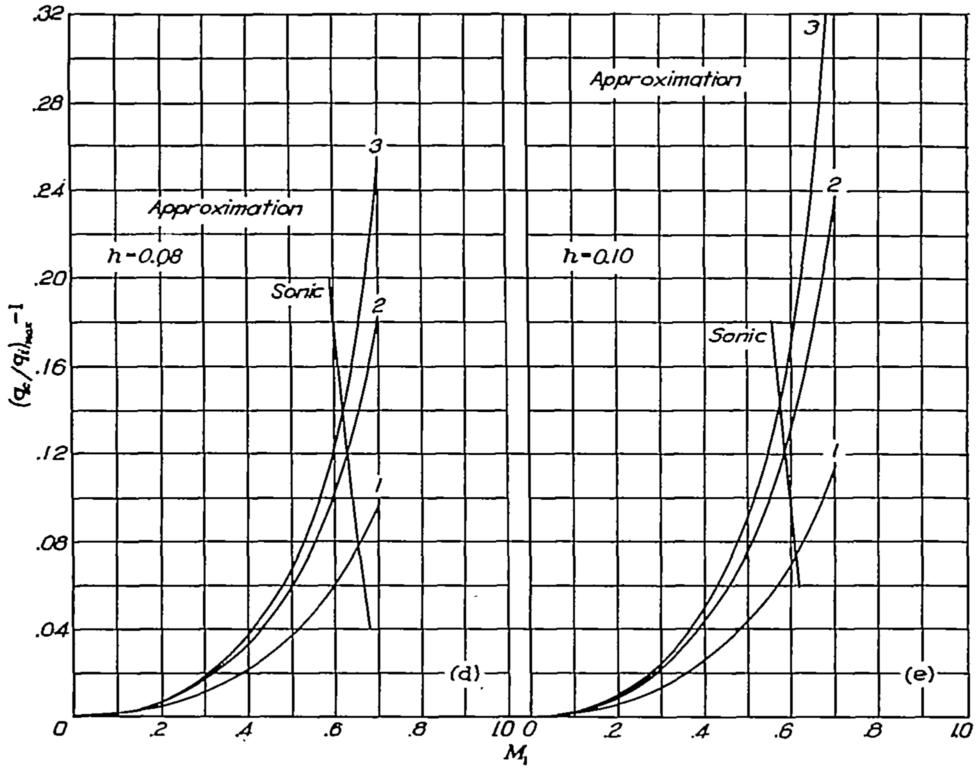

FIGURE 4.—Concluded.

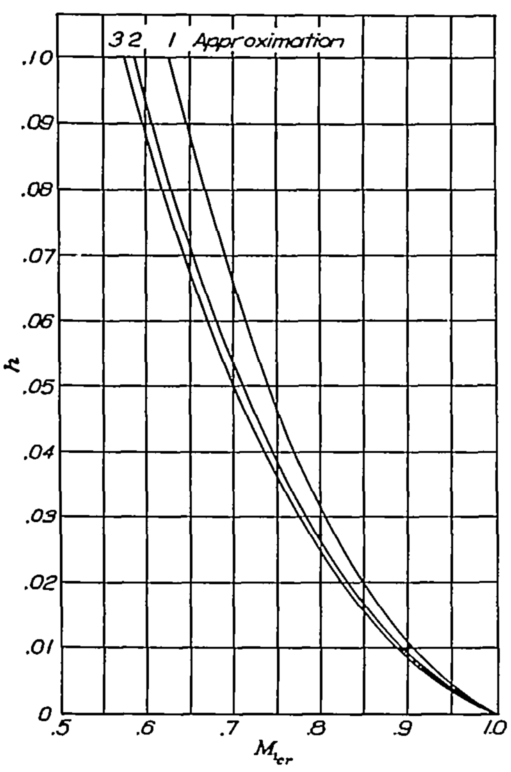

FIGURE 5.—Critical stream Mach number as a function of camber coefficient.

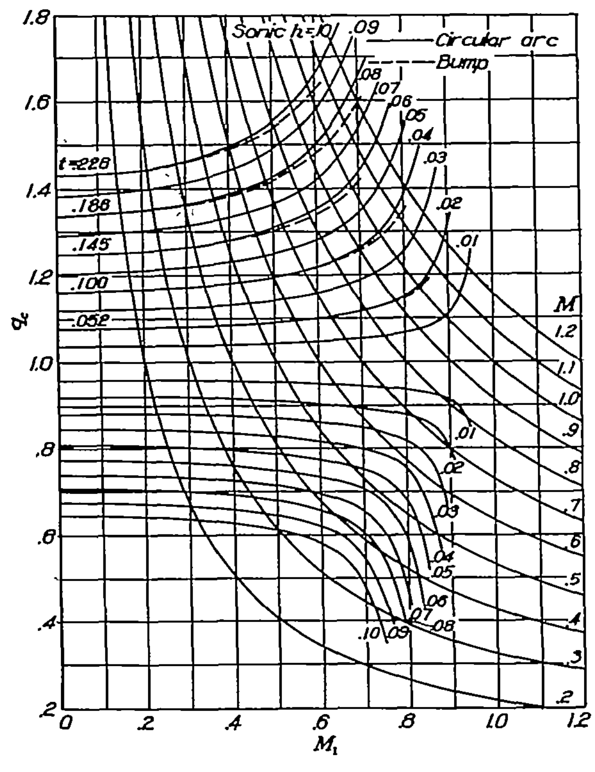

FIGURE 6.—Maximum and minimum velocities as functions of stream Mach number.

In general, the velocity q at the surface of the circular arc profile may be written as follows:

$$q=1+a_1h\sin\vartheta+h^2(a_2+a_3\cos 2\vartheta)$$
$$+h^3(a_4\sin\vartheta+a_5\sin 3\vartheta)+\ .\ .\ . \tag{25}$$

where, from equation (18),

$$a_1=\frac{4}{\beta}$$
$$a_2=-2+\frac{2}{\beta^4}+(\gamma-1)\left(\frac{1-\beta^2}{\beta^2}\right)^2$$
$$a_3=-\frac{4}{\beta^4}-2(\gamma-1)\left(\frac{1-\beta^2}{\beta^2}\right)^3$$
$$a_4=4\left[-\frac{2}{\beta}+G_1(0)+2G_2(0)\right]$$
$$a_5=8\left[-\frac{2}{\beta}+2\beta(2D+3)+G_2(0)\right]$$

Values of a_1, a_2, a_3, a_4, and a_5 for various values of the stream Mach number M_1 are given in table VIII. As an example of the behavior of the velocity distribution over a circular arc profile as the stream Mach number is varied, the case of $h=0.05$ with $M_1=0.3$, 0.5, and 0.7 is calculated and compared with the incompressible case. The calculated values of the velocity at the upper and lower surfaces of the circular arc profile, $h=0.05$, for the various values of M_1 are given in table IX and the corresponding velocity-distribution curves are shown in figure 7.

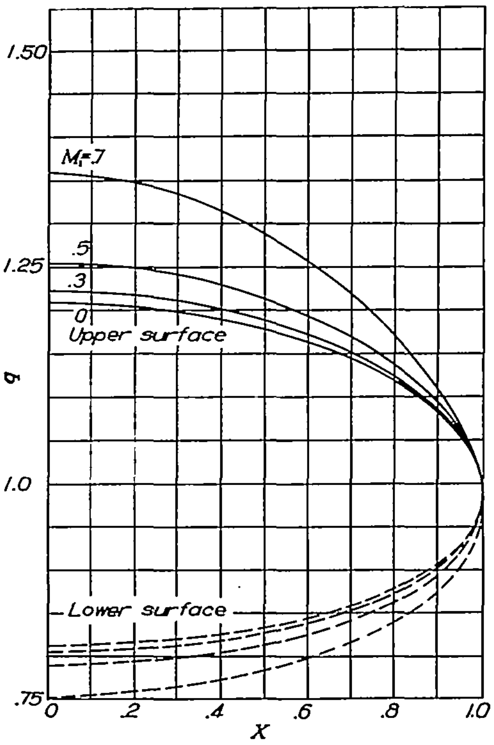

FIGURE 7.—Velocity distribution at upper and lower surfaces of circular arc profile, $h=0.05$, for various values of stream Mach number.

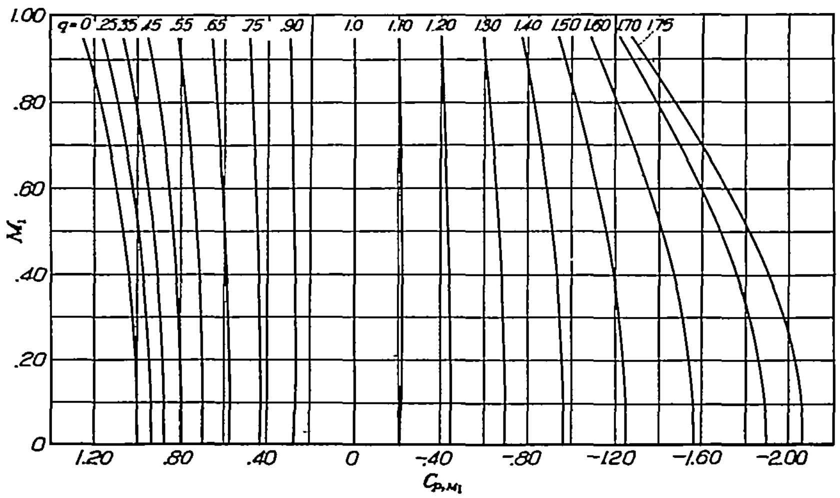

FIGURE 8.—Pressure coefficient C_{p,M_1} against stream Mach number M_1 for various values of fluid velocity q.

The pressure coefficient.—In the case of a uniform flow past a fixed boundary, the pressure coefficient is defined as

$$C_{p,M_1}=\frac{p-p_1}{\frac{1}{2}\rho_1U^2}$$

From the third of equations (6) it follows easily that

$$C_{p,M_1}=\frac{2}{\gamma M_1^2}\left\{-1+\left[1+\frac{1}{2}(\gamma-1)M_1^2(1-q^2)\right]^{\frac{\gamma}{\gamma-1}}\right\} \tag{26}$$

For the incompressible case, $M_1=0$,

$$C_{p,0}=1-q^2$$

For the sonic case, $q=q_{cr}$,

$$(C_{p,M_1})_{cr}=\frac{2}{\gamma M_1^2}\left\{-1+\left[\frac{2+(\gamma-1)M_1^2}{\gamma+1}\right]^{\frac{\gamma}{\gamma-1}}\right\}$$

For the limiting case of absolute vacuum, $M=\infty$ and

$$q=\left\{\frac{\frac{\gamma-1}{2}+\frac{1}{M_1^2}}{\frac{\gamma-1}{2}}\right\}^{1/2},$$

$$(C_{p,M_1})_{vac}=-\frac{2}{\gamma M_1^2}$$

Table X gives associated values of the velocity and the pressure coefficient C_{p,M_1} for various values of the stream Mach number M_1, and figure 8 shows the corresponding graphs. By means of table X and figure 8, the velocity readings from figures 6 and 7 can be replaced by the corresponding pressure coefficients.

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APPENDIX A

DETERMINATION OF THE COMPLEX POTENTIAL FUNCTION W

THE INCOMPRESSIBLE FLOW PAST A CIRCULAR ARC PROFILE

Consider the mapping of a circle C' in the Z'-plane into a circular arc C in the Z-plane. (See fig. 1.) If the center is at $(0,m)$ on the Y'-axis and the circle passes through the points $(a,0)$ and $(-a,0)$ on the X'-axis, then the Joukowski transformation

$$Z=Z'+\frac{a^2}{Z'} \tag{A1}$$

maps the circle C' in the Z'-plane into a circular arc C in the Z-plane. The equation of the circular arc is

$$X^2+\left(Y+\frac{a^2-m^2}{m}\right)^2=\left(\frac{m^2+a^2}{m}\right)^2 \tag{A2}$$

The parts of the circle C' lying above and below the X'-axis correspond, respectively, to the upper and lower surfaces of the circular arc C. The end points A and B of the circular arc are the points

$$X=\pm 2a$$

and the maximum ordinate is

$$Y=2a \tan \delta$$

$$=2m$$

The camber coefficient h is defined as the ratio of the maximum ordinate to the chord, or

$$h=\frac{2m}{4a}$$

$$=\frac{1}{2} \tan \delta \tag{A3}$$

The complex potential of the flow past a circular cylinder of radius R fixed in a uniform flow of velocity U at zero angle of attack and with a circulation Γ is given by

$$w=-U\left(Z''+\frac{R^2}{Z''}\right)-\frac{i\Gamma}{2\pi} \log \frac{Z''}{R} \tag{A4}$$

where

$$Z''=Z'-ia \tan \delta$$

For the purpose of the present paper the circulation Γ must be so chosen that the stagnation points on the circle C' lie at the points $X'=\pm a$ corresponding to the leading and trailing edges of the circular arc C; that is,

$$\Gamma=8\pi Uah$$

$$=4\pi UR \sin \delta \tag{A5}$$

With this value of the circulation inserted in equation (A4) and with Z'' replaced by $Re^{i\theta}$, the complex velocity at the surface of the circular arc C becomes

$$\frac{dw}{dZ}=2iUe^{-i\theta} \frac{\sin \theta+\sin \delta}{1-2ie^{i\theta} \sin \delta-e^{2i\theta}} (e^{i\theta}+i \sin \delta)^2$$

The magnitude of the velocity is

$$q=\left(\frac{dw}{dZ}\frac{d\bar{w}}{d\bar{Z}}\right)^{1/2}$$

$$=U(1+2 \sin \theta \sin \delta+\sin^2 \delta) \tag{A6}$$

It is recalled that the upper surface of the circular arc is traversed in a clockwise sense as θ goes from $-\delta$ to $\pi+\delta$ and the lower surface, as θ goes from $-(\pi-\delta)$ to $-\delta$. The velocity at the nose or tail is then given by

$$q_{nose}=q_{tail}=U \cos^2\delta$$

The maximum and minimum velocities occur at $\theta=\frac{\pi}{2}$ and at $\theta=-\frac{\pi}{2}$, respectively, and are given by

$$\left.\begin{aligned} q_{max}&=U(1+\sin \delta)^2 \\ q_{min}&=U(1-\sin \delta)^2 \end{aligned}\right\} \tag{A7}$$

EQUATION OF CIRCULAR ARC AS POWER SERIES IN h

The equation of the circular arc, obtained from equation (A2) for the entire circle, is

$$Y=2m-r+(r^2-X^2)^{1/2} \tag{A8}$$

where $r=\frac{m^2+a^2}{m}$ is the radius of the circle. Expansion of the radical in equation (A8) according to powers of X/r yields

$$Y=2m-\frac{1}{2}\frac{X^2}{r}-\frac{1}{8}\frac{X^4}{r^3}-\frac{1}{16}\frac{X^6}{r^5}- \dots \tag{A9}$$

By use of $h=\frac{m}{2a}$

$$\frac{a}{r}=\frac{2h}{1+4h^2}$$

$$=2h-8h^3+32h^5- \dots$$

Then equation (A9) becomes

$$Y=\left(4a-\frac{X^2}{a}\right)h+\left(4\frac{X^2}{a}-\frac{X^4}{a^3}\right)h^3$$
$$+\left(-16\frac{X^2}{a}+12\frac{X^4}{a^3}-2\frac{X^6}{a^5}\right)h^5+ \dots \tag{A10}$$

Now, put $\frac{X}{2a}=\cos \vartheta$ and replace $\frac{X}{2a}$ and $\frac{Y}{2a}$ by X and Y, respectively. Equation (A10) then becomes

$$Y=2h \sin^2 \vartheta+2h^3 \sin^2 2\vartheta+8h^5 \sin^2 2\vartheta \cos 2\vartheta+ \dots \tag{A11}$$

and

$$\frac{dY}{dX}=-4h \cos \vartheta-16h^3 \cos \vartheta \cos 2\vartheta$$
$$-16h^5 \cos \vartheta (1+3 \cos 4\vartheta)- \dots \tag{A12}$$

EQUATION OF w AS A POWER SERIES IN h

Consider equation (A4) with $\Gamma=8\pi Uah$ and $R^2=a^2(1+4h^2)$.

Then

$$w=-U\left(Z''+a^2\frac{1+4h^2}{Z''}\right)-4iUah\log\frac{Z''}{R} \tag{A13}$$

Now

$$Z''=Z'-ia\tan\delta$$

$$=Z'-2iah$$

Then by expanding the right-hand side of equation (A13) according to powers of h and replacing Z' by $\frac{Z+(Z^2-4a^2)^{1/2}}{2}$ obtained from the Joukowski transformation (A1), it follows that

$$w=-UZ+2iaUh\left\{1-\frac{[Z-(Z^2-4a^2)^{1/2}]^2}{4a^2}\right.$$
$$\left.-2\log\frac{Z+(Z^2-4a^2)^{1/2}}{2}\right\}+\ldots$$

If $w/2aU$ and $Z/2a$ are written, respectively, w and Z, then

$$w=-Z+ih\{1-[Z-(Z^2-1)^{1/2}]^2$$
$$-2\log[Z+(Z^2-1)^{1/2}]\}+\ldots \tag{A14}$$

From equation (A14) for w and a corresponding equation for the complex conjugate $\overline{w}$, the nondimensional velocity potential becomes

$$\phi=-X+\frac{ih}{2}\left\{-[Z-(Z^2-1)^{1/2}]^2\right.$$
$$\left.+[\overline{Z}-(\overline{Z}^2-1)^{1/2}]^2-2\log\frac{Z+(Z^2-1)^{1/2}}{\overline{Z}+(\overline{Z}^2-1)^{1/2}}\right\}+\ldots \tag{A15}$$

APPENDIX B

DETERMINATION OF THE FIRST APPROXIMATION ϕ_1

By means of transformation (14), equation (11) for ϕ_1 becomes

$$\frac{\partial^2\phi_1}{\partial x^2}+\frac{\partial^2\phi_1}{\partial y^2}=0 \tag{B1}$$

A comparison of the expressions for ϕ given by equations (9) and (A15) suggests the assumption

$$\phi_1=k\{[z-(z^2-1)^{1/2}]^2-[\overline{z}-(\overline{z}^2-1)^{1/2}]^2$$
$$+2\log\frac{z+(z^2-1)^{1/2}}{\overline{z}+(\overline{z}^2-1)^{1/2}}\right\} \tag{B2}$$

where $z=x+iy$, $\overline{z}=x-iy$, and k is an arbitrary constant. Since this expression for ϕ_1 is the sum of a function of z only and a function of $\overline{z}$ only, it satisfies Laplace's equation (B1). The arbitrary constant k is to be determined from the boundary condition

$$\frac{\partial\phi}{\partial X}\frac{dY}{dX}=\frac{\partial\phi}{\partial Y}$$

or

$$\frac{\partial\phi}{\partial x}\frac{dy}{dx}=\beta^2\frac{\partial\phi}{\partial y} \tag{B3}$$

The expression for ϕ, insofar as the first power in h is concerned, is

$$\phi=-x-h\phi_1$$

and, to the first power in h, from equation (A12),

$$\frac{dy}{dx}=-4\beta h\cos\vartheta$$

The boundary condition, equation (B3), then becomes

$$4\beta h\cos\vartheta=-\beta^2h\frac{\partial\phi_1}{\partial y}$$

$$=-i\beta^2h\left(\frac{\partial\phi_1}{\partial z}-\frac{\partial\phi_1}{\partial\overline{z}}\right)$$

$$=2ikh\beta^2\left\{\frac{[z-(z^2-1)^{1/2}]^2-1}{(z^2-1)^{1/2}}+\frac{[\overline{z}-(\overline{z}^2-1)^{1/2}]^2-1}{(\overline{z}^2-1)^{1/2}}\right\} \tag{B4}$$

By definition $x=\cos\vartheta$, and from equation (A11), to the first power in h, $y=2\beta h\sin^2\vartheta$. Hence, to the first power in h,

$$z=\cos\vartheta+2i\beta h\sin^2\vartheta$$

$$\overline{z}=\cos\vartheta-2i\beta h\sin^2\vartheta$$

$$z^2=\cos^2\vartheta+4i\beta h\sin^2\vartheta\cos\vartheta$$

$$\overline{z}^2=\cos^2\vartheta-4i\beta h\sin^2\vartheta\cos\vartheta$$

$$(z^2-1)^{1/2}=i\sin\vartheta(1-2i\beta h\cos\vartheta)$$

$$(\overline{z}^2-1)^{1/2}=-i\sin\vartheta(1+2i\beta h\cos\vartheta)$$

Then

$$4\beta h\cos\vartheta=-2ikh\beta^2\frac{e^{2i\vartheta}-e^{-2i\vartheta}}{i\sin\vartheta}$$

$$=-4ikh\beta^2\frac{\sin 2\vartheta}{\sin\vartheta}$$

$$=-8ikh\beta^2\cos\vartheta$$

or

$$k=\frac{i}{2\beta}$$

The expression for the first approximation of ϕ is then

$$\phi=-x-\frac{ih}{2\beta}\left\{\left[z-(z^2-1)^{1/2}\right]^2-\left[\overline{z}-(\overline{z}^2-1)^{1/2}\right]^2\right.$$
$$\left.+2\log\frac{z+(z^2-1)^{1/2}}{\overline{z}+(\overline{z}^2-1)^{1/2}}\right\} \tag{B5}$$

This expression for ϕ can be simplified considerably by introducing elliptic coordinates ξ and η. Thus, let

$$z=\cosh\zeta \tag{B6}$$

where

$$\zeta=\xi+i\eta$$

Then

$$x+iy=\cosh(\xi+i\eta)$$
$$=\cosh\xi\cos\eta+i\sinh\xi\sin\eta$$

so that

$$\left.\begin{aligned} x&=\cosh\xi\cos\eta \\ y&=\sinh\xi\sin\eta \end{aligned}\right\} \tag{B7}$$

Equation (B5) can then be written

$$\phi=-\frac{1}{2}(\cosh\zeta+\cosh\bar{\zeta})-\frac{ih}{2\beta}(e^{-2\zeta}-e^{-2\bar{\zeta}}+2\log e^{\zeta-\bar{\zeta}}) \tag{B8}$$

or

$$\phi=-\cosh\xi\cos\eta-\frac{h}{\beta}(e^{-2\xi}\sin 2\eta-2\eta) \tag{B9}$$

From a comparison of equations (A15) and (B5) note that, if Γ_i and Γ_c denote the circulation in the incompressible case and the compressible case, then

$$\begin{aligned} \frac{\Gamma_c}{\Gamma_i}&=\frac{1}{\beta} \\ &=\frac{1}{(1-M_1^2)^{1/2}} \end{aligned} \tag{B10}$$

Equation (B10) is the well-known Prandtl-Glauert rule connecting the circulations (or lifts) in the incompressible and compressible cases.

In order to utilize equation (B9) for the calculations, the equations of transformation (B7) must be inverted. Thus,

$$\left.\begin{aligned} \frac{x^2}{\cosh^2\xi}+\frac{y^2}{\sinh^2\xi}&=1 \\ \frac{x^2}{\cos^2\eta}-\frac{y^2}{\sin^2\eta}&=1 \end{aligned}\right\} \tag{B11}$$

From equations (B11),

$$\left.\begin{aligned} 2\sinh^2\xi&=-b+(b^2+4y^2)^{1/2} \\ 2\sin^2\eta&=b+(b^2+4y^2)^{1/2} \end{aligned}\right\} \tag{B12}$$

where

$$b=1-(x^2+y^2)$$

By means of transformation (14),

$$2\sinh^2\xi=-b+(b^2+4\beta^2Y^2)^{1/2}$$

$$2\sin^2\eta=b+(b^2+4\beta^2Y^2)^{1/2}$$

where

$$b=1-(X^2+\beta^2Y^2)$$

In terms of the complex variables ζ and $\bar{\zeta}$, the velocity components in the direction of the coordinate axes are

$$\left.\begin{aligned} u&=\frac{\partial\phi}{\partial X}=\frac{1}{\sinh\zeta}\frac{\partial\phi}{\partial\zeta}+\frac{1}{\sinh\bar{\zeta}}\frac{\partial\phi}{\partial\bar{\zeta}} \\ v&=\frac{\partial\phi}{\partial Y}=i\beta\left(\frac{1}{\sinh\zeta}\frac{\partial\phi}{\partial\zeta}-\frac{1}{\sinh\bar{\zeta}}\frac{\partial\phi}{\partial\bar{\zeta}}\right) \end{aligned}\right\} \tag{B13}$$

Let ϕ be given by equation (B8); then,

$$\left.\begin{aligned} u&=-1-\frac{4h}{\beta}e^{-\xi}\sin\eta \\ v&=4he^{-\xi}\cos\eta \end{aligned}\right\} \tag{B14}$$

Now, to the first power in h, at the boundary,

$$\xi=0$$

$$\eta=\vartheta$$

Hence, if q_c and q_i denote the magnitudes of the velocity at the surface of the circular arc profile for the compressible and the incompressible cases, respectively, then

$$\left.\begin{aligned} q_c&=1+\frac{4h}{\beta}\sin\vartheta \\ q_i&=1+4h\sin\vartheta \end{aligned}\right\} \tag{B15}$$

or, when $h\sin\vartheta$ is eliminated,

$$\frac{q_c}{q_i}=\frac{1}{\beta}-\left(\frac{1}{\beta}-1\right)\frac{1}{q_i} \tag{B16}$$

where

$$\beta=(1-M_1^2)^{1/2}$$

Equation (B16) represents the velocity correction formula for the Prandtl-Glauert approximation. Equations (B15) can also be written as follows:

$$\frac{q_c-1}{q_i-1}=\frac{1}{\beta} \tag{B17}$$

Since the Prandtl-Glauert approximation is strictly true for infinitesimal disturbances to the uniform stream, equation (B17) may be replaced by the differential coefficient (from reference 11)

$$\left(\frac{dq_c}{dq_i}\right)_{q_i=1}=\frac{1}{\beta} \tag{B18}$$

APPENDIX C

DETERMINATION OF THE SECOND APPROXIMATION ϕ_2

By means of transformation (14), the symbolic relations,

$$\left.\begin{aligned}
\frac{\partial}{\partial x}&=\frac{\partial}{\partial z}+\frac{\partial}{\partial \bar{z}}\\
\frac{\partial^2}{\partial x^2}&=\frac{\partial^2}{\partial z^2}+2\,\frac{\partial^2}{\partial z\partial \bar{z}}+\frac{\partial^2}{\partial \bar{z}^2}\\
\frac{\partial}{\partial y}&=i\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial \bar{z}}\right)\\
\frac{\partial^2}{\partial y^2}&=-\frac{\partial^2}{\partial z^2}+2\,\frac{\partial^2}{\partial z\partial \bar{z}}-\frac{\partial^2}{\partial \bar{z}^2}\\
\frac{\partial^2}{\partial x\partial y}&=i\left(\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial \bar{z}^2}\right)
\end{aligned}\right\} \tag{C1}$$

and the equation of transformation (B6), $z=\cosh \zeta$, or $\bar{z}=\cosh \bar{\zeta}$, differential equation (12) for ϕ_2 can be expressed in terms of the complex variables ζ and $\bar{\zeta}$ as follows:

$$\begin{aligned}\frac{4}{\sinh \zeta \sinh \bar{\zeta}}\,\frac{\partial^2\phi_2}{\partial\zeta\,\partial\bar{\zeta}}=\frac{1-\beta^2}{\beta^2}\Bigg\{&[(\gamma+1)-(\gamma-1)\beta^2]\left(\frac{1}{\sinh \zeta}\,\frac{\partial\phi_1}{\partial\zeta}+\frac{1}{\sinh \bar{\zeta}}\,\frac{\partial\phi_1}{\partial\bar{\zeta}}\right)\left(\frac{1}{\sinh^2\zeta}\,\frac{\partial^2\phi_1}{\partial\zeta^2}-\frac{\cosh\zeta}{\sinh^3\zeta}\,\frac{\partial\phi_1}{\partial\zeta}+\frac{1}{\sinh^2\bar{\zeta}}\,\frac{\partial^2\phi_1}{\partial\bar{\zeta}^2}-\frac{\cosh\bar{\zeta}}{\sinh^3\bar{\zeta}}\,\frac{\partial\phi_1}{\partial\bar{\zeta}}\right)\\
&-2\beta^2\left(\frac{1}{\sinh \zeta}\,\frac{\partial\phi_1}{\partial\zeta}-\frac{1}{\sinh \bar{\zeta}}\,\frac{\partial\phi_1}{\partial\bar{\zeta}}\right)\left(\frac{1}{\sinh^2\zeta}\,\frac{\partial^2\phi_1}{\partial\zeta^2}-\frac{\cosh\zeta}{\sinh^3\zeta}\,\frac{\partial\phi_1}{\partial\zeta}-\frac{1}{\sinh^2\bar{\zeta}}\,\frac{\partial^2\phi_1}{\partial\bar{\zeta}^2}+\frac{\cosh\bar{\zeta}}{\sinh^3\bar{\zeta}}\,\frac{\partial\phi_1}{\partial\bar{\zeta}}\right)\Bigg\}\end{aligned} \tag{C2}$$

When the expression for ϕ_1 obtained from equation (B8),

$$\phi_1=\frac{i}{2\beta}\,[e^{-2\zeta}-e^{-2\bar{\zeta}}+2(\zeta-\bar{\zeta})]$$

is introduced into the right-hand side of equation (C2), the following differential equation for ϕ_2 results:

$$\frac{\partial^2\phi_2}{\partial\zeta\,\partial\bar{\zeta}}=\frac{1-\beta^2}{\beta^4}\{[(\gamma+1)-(\gamma-1)\beta^2](e^{-\zeta}-e^{-\bar{\zeta}})(e^{-\zeta}\sinh\bar{\zeta}-e^{-\bar{\zeta}}\sinh\zeta)-8\beta^2(e^{-\zeta}-e^{-\bar{\zeta}})(e^{-\zeta}\sinh\bar{\zeta}+e^{-\bar{\zeta}}\sinh\zeta)\}$$

Finally, by putting $\zeta=\xi+i\eta$ and $\bar{\zeta}=\xi-i\eta$:

$$\frac{\partial^2\phi_2}{\partial\xi^2}+\frac{\partial^2\phi_2}{\partial\eta^2}=4\frac{1-\beta^2}{\beta^4}(\{-[(\gamma+1)-(\gamma-3)\beta^2]e^{-\xi}+4\beta^2e^{-3\xi}\}\cos\eta+(\gamma+1)(1-\beta^2)e^{-\xi}\cos 3\eta) \tag{C3}$$

The right-hand side of equation (C3) suggests a solution of the form

$$\phi_2=F_1(\xi)\cos\eta+F_3(\xi)\cos 3\eta \tag{C4}$$

By substituting this expression for ϕ_2 into equation (C3) and by equating the coefficients of $\cos\eta$ and $\cos 3\eta$ to zero, the following differential equations for $F_1(\xi)$ and $F_3(\xi)$ are obtained:

$$\frac{d^2F_1}{d\xi^2}-F_1=4\frac{1-\beta^2}{\beta^4}\{-[(\gamma+1)-(\gamma-3)\beta^2]e^{-\xi}+4\beta^2e^{-3\xi}\} \tag{C5}$$

$$\frac{d^2F_3}{d\xi^2}-9F_3=4(\gamma+1)\left(\frac{1-\beta^2}{\beta^2}\right)^2e^{-\xi} \tag{C6}$$

The solutions of these equations are

$$F_1=2\frac{1-\beta^2}{\beta^4}\{[(\gamma+1)-(\gamma-3)\beta^2]\xi e^{-\xi}+\beta^2e^{-3\xi}+A_1e^{-\xi}\} \tag{C7}$$

$$F_3=-\frac{1}{2}(\gamma+1)\left(\frac{1-\beta^2}{\beta^2}\right)^2(e^{-\xi}+A_3e^{-3\xi}) \tag{C8}$$

where A_1 and A_3 are arbitrary constants to be determined by the boundary condition at the surface of the profile. The other two arbitrary constants are taken equal to zero since F_1 and F_3 must vanish at infinity.

In terms of the variables ξ and η, the boundary condition (B3) takes the form

$$\left(\sinh\xi\cos\eta\,\frac{\partial\phi}{\partial\xi}-\cosh\xi\sin\eta\,\frac{\partial\phi}{\partial\eta}\right)\frac{dy}{dx}=\beta^2\left(\cosh\xi\sin\eta\,\frac{\partial\phi}{\partial\xi}+\sinh\xi\cos\eta\,\frac{\partial\phi}{\partial\eta}\right) \tag{C9}$$

where the velocity potential ϕ has the form

$$\phi=-\cosh \xi \cos \eta-\frac{h}{\beta}(e^{-2\xi} \sin 2\eta-2\eta)$$
$$-h^2(F_1 \cos \eta+ F_3 \cos 3\eta+\Gamma_2\eta) \qquad \text{(C10)}$$

and where Γ_2 is an arbitrary circulation to be determined by the Kutta condition at the trailing edge of the circular arc profile.

In order to make use of the boundary equation (C9), the various functions of ξ and η appearing in equation (C10) must be expressed as functions of ϑ evaluated at the boundary. From equations (A11) and (A12), the boundary and its slope are now given by

$$y=2\beta h \sin^2 \vartheta+8\beta h^3 \sin^2 \vartheta \cos^2 \vartheta+ \ . \ . \ .$$
$$\frac{dy}{dx}=-4\beta h \cos \vartheta- \ . \ . \ .$$

At the boundary then, with $x=\cos \vartheta$, when powers of h above the second are neglected,

$$b=1-(x^2+y^2)$$
$$=\sin^2 \vartheta-4\beta^2h^2 \sin^4 \vartheta$$

Then, from equations (B12)

$$\sin^2 \eta=\sin^2 \vartheta(1+4\beta^2h^2 \cos^2 \vartheta)$$
$$\cos^2 \eta=\cos^2 \vartheta(1-4\beta^2h^2 \sin^2 \vartheta)$$
$$\sin \eta=\sin \vartheta(1+2\beta^2h^2 \cos^2 \vartheta)$$
$$\cos \eta=\cos \vartheta(1-2\beta^2h^2 \sin^2 \vartheta)$$
$$\sinh^2 \xi=4\beta^2h^2 \sin^2 \vartheta$$
$$\cosh^2 \xi=1+4\beta^2h^2 \sin^2 \vartheta$$
$$\sinh \xi=2\beta h \sin \vartheta$$
$$\cosh \xi=1+2\beta^2h^2 \sin^2 \vartheta$$
$$e^{-\xi}=1-2\beta h \sin \vartheta+2\beta^2h^2 \sin^2 \vartheta$$
$$\xi=2\beta h \sin \vartheta$$

When these expressions, with equations (C7), (C8), and (C10), are utilized in the boundary equation (C9), the following results are obtained:

$$\left.\begin{aligned} 2\,\frac{1-\beta^2}{\beta^4}A_1&=\frac{6}{\beta^2}+2(\gamma+1)\left(\frac{1-\beta^2}{\beta^2}\right)^2 \\ -\frac{\gamma+1}{2}\left(\frac{1-\beta^2}{\beta^2}\right)^2 A_3&=-\frac{2}{3}\,\frac{2+\beta^2}{\beta^2}+\frac{1}{6}(\gamma+1)\left(\frac{1-\beta^2}{\beta^2}\right)^2 \end{aligned}\right\} \qquad \text{(C11)}$$

The value of the arbitrary constant Γ_2 is determined in the following way. The magnitude of the velocity, when terms containing powers of h higher than the second are neglected, is given by

$$q=1+h\,\frac{\partial\phi_1}{\partial x}+h^2\left[\frac{1}{2}\,\beta^2\left(\frac{\partial\phi_1}{\partial y}\right)^2+\frac{\partial\phi_2}{\partial x}\right]+ \ . \ . \ .$$

or, in the variables ξ and η,

$$q=1+\frac{2h}{\cosh 2\xi-\cos 2\eta}\left(\sinh \xi \cos \eta\,\frac{\partial\phi_1}{\partial\xi}-\cosh \xi \sin \eta\,\frac{\partial\phi_1}{\partial\eta}\right)$$
$$+\frac{2h^2}{\cosh 2\xi-\cos 2\eta}\left(\sinh \xi \cos \eta\,\frac{\partial\phi_2}{\partial\xi}-\cosh \xi \sin \eta\,\frac{\partial\phi_3}{\partial\eta}\right)$$
$$+\frac{2\beta^2h^2}{(\cosh 2\xi-\cos 2\eta)^2}\left(\cosh \xi \sin \eta\,\frac{\partial\phi_1}{\partial\xi}+\sinh \xi \cos \eta\,\frac{\partial\phi_1}{\partial\eta}\right)^2+\ldots \qquad \text{(C12)}$$

From equations (B9) and (C10),

$$\phi_1=\frac{1}{\beta}(e^{-2\xi} \sin 2\eta-2\eta)$$

and

$$\phi_2=F_1 \cos \eta+F_3 \cos 3\eta+\Gamma_2\eta \qquad \text{(C13)}$$

where F_1 and F_3 are obtained from equations (C7), (C8), and (C11). At the boundary, equation (C12) becomes

$$q=1+\frac{4h}{\beta}\sin\vartheta+h^2\left\{-2-\frac{2}{\beta^4}-(\gamma-1)\left(\frac{1-\beta^2}{\beta^2}\right)^2+4\left[\frac{2}{\beta^4}+(\gamma-1)\left(\frac{1-\beta^2}{\beta^2}\right)^2\right]\sin^2\vartheta-\frac{\Gamma_2}{\sin\vartheta}\right\}+\ldots \qquad \text{(C14)}$$

The Kutta condition at the trailing edge $\vartheta=\pi$ requires that the velocity be finite; therefore, Γ_2 must be zero and the compressibility effect on the circulation (or lift) is again given by the Prandtl-Glauert rule.

If q_c and q_i denote the magnitudes of the velocity at the boundary in the compressible and incompressible cases, respectively, the velocity correction formula is

$$\frac{q_c}{q_i}=\frac{1+\frac{4h}{\beta}\sin\vartheta+h^2\left\{-2-\frac{2}{\beta^4}-(\gamma-1)\left(\frac{1-\beta^2}{\beta^2}\right)^2+4\left[\frac{2}{\beta^4}+(\gamma-1)\left(\frac{1-\beta^2}{\beta^2}\right)^2\right]\sin^2\vartheta\right\}}{1+4h\sin\vartheta-4h^2\cos 2\vartheta} \qquad \text{(C15)}$$

where $0\leqq \vartheta \leqq \pi$ for the upper side of the circular arc and $-\pi\leqq\vartheta\leqq 0$ for the lower side of the circular arc.

Then, for the leading or trailing edge, $\vartheta=0$ or $\vartheta=\pi$,

$$\frac{q_c}{q_i}=\frac{1-h^2\left[\left(\frac{1}{\beta^2}+1\right)^2+\gamma\left(\frac{1}{\beta^2}-1\right)^2\right]}{1-4h^2} \qquad \text{(C16)}$$

For the position of maximum velocity, $\vartheta=\frac{\pi}{2}$,

$$\frac{q_c}{q_i}=\frac{1+\frac{4h}{\beta}+h^2\left[-8+3\left(\frac{1}{\beta^2}+1\right)^2+3\gamma\left(\frac{1}{\beta^2}-1\right)^2\right]}{(1+2h)^2} \qquad \text{(C17)}$$

For the position of minimum velocity, $\vartheta=-\frac{\pi}{2}$,

$$\frac{q_c}{q_i}=\frac{1-\frac{4h}{\beta}+h^2\left[-8+3\left(\frac{1}{\beta^2}+1\right)^2+3\gamma\left(\frac{1}{\beta^2}-1\right)^2\right]}{(1-2h)^2} \qquad \text{(C18)}$$

APPENDIX D

DETERMINATION OF THE THIRD APPROXIMATION ϕ_3

The differential equation (13) for ϕ_3 can be expressed in terms of z and $\bar{z}$ by means of transformation (14) and the symbolic relations (C1). Thus

$$\begin{aligned} 4\phi_{3z\bar{z}}=\frac{1-\beta^2}{\beta^2}\Big\{&\frac{1}{2}[(\gamma+1)(1-\beta^2)+4\beta^2](\phi_{1zz}+\phi_{1\bar{z}\bar{z}})(\phi_{1z}^2+\phi_{1\bar{z}}^2)+(\gamma+1)(1-\beta^4)(\phi_{1zz}+\phi_{1\bar{z}\bar{z}})\phi_{1z}\phi_{1\bar{z}} \\ &+[(\gamma+1)(1-\beta^2)+2\beta^2][(\phi_{1zz}+\phi_{1\bar{z}\bar{z}})(\phi_{2z}+\phi_{2\bar{z}})+(\phi_{1z}+\phi_{1\bar{z}})(\phi_{2zz}+\phi_{2\bar{z}\bar{z}})] \\ &+2[(\gamma+1)(1+\beta^2)-2\beta^2](\phi_{1z}+\phi_{1\bar{z}})\phi_{2z\bar{z}} \\ &-2\beta^2[(\phi_{1z}^2-\phi_{1\bar{z}}^2)(\phi_{1zz}-\phi_{1\bar{z}\bar{z}})+(\phi_{2z}-\phi_{2\bar{z}})(\phi_{1zz}-\phi_{1\bar{z}\bar{z}})+(\phi_{1z}-\phi_{1\bar{z}})(\phi_{2zz}-\phi_{2\bar{z}\bar{z}})]\Big\} \end{aligned} \tag{D1}$$

where, for example,

$$\phi_{3z\bar{z}}\equiv\frac{\partial^2\phi_3}{\partial z\partial\bar{z}}$$

The expression for ϕ_1, obtained from equation (B5), is now

$$\phi_1=\frac{i}{2\beta}\left\{[z-(z^2-1)^{1/2}]^2-[\bar{z}-(\bar{z}^2-1)^{1/2}]^2+2\log\frac{z+(z^2-1)^{1/2}}{\bar{z}+(\bar{z}^2-1)^{1/2}}\right\} \tag{D2}$$

Introduce new complex variables λ and $\bar{\lambda}$, where

$$\left.\begin{aligned} \lambda&=z+(z^2-1)^{1/2} & \frac{1}{\lambda}&=z-(z^2-1)^{1/2} \\ \bar{\lambda}&=\bar{z}+(\bar{z}^2-1)^{1/2} & \frac{1}{\bar{\lambda}}&=\bar{z}-(\bar{z}^2-1)^{1/2} \end{aligned}\right\} \tag{D3}$$

The relations between the complex variables λ and $\bar{\lambda}$ and the complex variables ζ and $\bar{\zeta}$, respectively, are

$$\left.\begin{aligned} \lambda&=e^{\zeta} & \bar{\lambda}&=e^{\bar{\zeta}} \\ \lambda\bar{\lambda}&=e^{2\xi} & \frac{\lambda}{\bar{\lambda}}&=e^{2i\eta} \end{aligned}\right\} \tag{D4}$$

Then

$$\phi_1=\frac{1}{2i\beta}\left(\frac{1}{\bar{\lambda}^2}-\frac{1}{\lambda^2}-2\log\frac{\lambda}{\bar{\lambda}}\right) \tag{D5}$$

Similarly, the expression for ϕ_2, obtained from equations (C4), (C7), (C8), and (C11), is

$$\begin{aligned} \phi_2=2(D-E)\frac{\lambda+\bar{\lambda}}{\lambda\bar{\lambda}}\log\lambda\bar{\lambda}+D\frac{\lambda+\bar{\lambda}}{\lambda^2\bar{\lambda}^2}+(-3C+D-5E)\frac{\lambda+\bar{\lambda}}{\lambda\bar{\lambda}} \\ +E\frac{\lambda^3+\bar{\lambda}^3}{\lambda^2\bar{\lambda}^2}+C\frac{\lambda^3+\bar{\lambda}^3}{\lambda^3\bar{\lambda}^3} \end{aligned} \tag{D6}$$

where

$$C=-1-\frac{2}{3}\frac{1-\beta^2}{\beta^2}+\frac{1}{12}(\gamma+1)\left(\frac{1-\beta^2}{\beta^2}\right)^2$$

$$D=\frac{1-\beta^2}{\beta^2}$$

$$E=-\frac{\gamma+1}{4}\left(\frac{1-\beta^2}{\beta^2}\right)^2$$

and

$$C-\frac{2}{3}D-\frac{1}{3}E=1$$

From equations (D5) and (D6) with the use of equation (D3), the following relations are obtained:

$$\phi_{1z}=-\frac{2}{i\beta}\frac{1}{\lambda}$$

$$\phi_{1zz}=\frac{4}{i\beta}\frac{1}{\lambda^2-1}$$

$$\begin{aligned} \phi_{2z}=&4(D-E)\left(\frac{\lambda+\lambda}{\bar{\lambda}}\frac{1}{\lambda^2-1}-\frac{1}{\lambda^2-1}\log\lambda\bar{\lambda}\right)-2D\frac{\lambda+2\bar{\lambda}}{\lambda^2\bar{\lambda}^2(\lambda^2-1)} \\ &+2(3C-D+5E)\frac{1}{\lambda^2-1}+2E\frac{\lambda^3-2\lambda^3}{\lambda^2\bar{\lambda}^2(\lambda^2-1)}-6C\frac{1}{\lambda^2(\lambda^2-1)} \end{aligned}$$

$$\begin{aligned} \phi_{2zz}=&8(D-E)\left[\frac{2\lambda^3}{(\lambda^2-1)^3}\log\lambda\bar{\lambda}-\frac{2\lambda^3(\lambda+\bar{\lambda})}{\bar{\lambda}(\lambda^2-1)^3}+\frac{\lambda(\lambda-\bar{\lambda})}{\bar{\lambda}(\lambda^2-1)^2}\right] \\ &+8D\frac{\lambda^3+3\lambda^2\bar{\lambda}-\lambda}{\bar{\lambda}^2(\lambda^2-1)^3}-8(3C-D+5E)\frac{\lambda^3}{(\lambda^2-1)^3} \\ &-8E\frac{\lambda^3-3\lambda^2\bar{\lambda}^3+\bar{\lambda}^3}{\bar{\lambda}^2(\lambda^2-1)^3}+24C\frac{2\lambda^2-1}{\lambda(\lambda^2-1)^3} \end{aligned}$$

$$\phi_{2z\bar{z}}=8D\left(\frac{1}{\lambda\bar{\lambda}}-1\right)\frac{\lambda+\bar{\lambda}}{(\lambda^2-1)(\bar{\lambda}^2-1)}-8E\frac{(\lambda+\lambda)(\lambda-\bar{\lambda})^2}{\lambda\bar{\lambda}(\lambda^2-1)(\bar{\lambda}^2-1)}$$

and expressions for the corresponding conjugate complex quantities.

When the foregoing expressions are introduced into equation (D1), and when equations (D4) are used to express the various quantities in terms of the variables ξ and η, the following differential equation for ϕ_3 is obtained:

$$\begin{aligned}\frac{\partial^2\phi_3}{\partial\xi^2}+\frac{\partial^2\phi_3}{\partial\eta^2}=&[(A_2^2+B_2^2\xi)e^{-2\xi}+(A_4^2+B_4^2\xi)e^{-4\xi}+(A_6^2+B_6^2\xi)e^{-6\xi}+A_8^2e^{-8\xi}+A_{10}^2e^{-10\xi}]\sin 2\eta\\ &+[(A_2^4+B_2^4\xi)e^{-2\xi}+(A_4^4+B_4^4\xi)e^{-4\xi}+(A_6^4+B_6^4\xi)e^{-6\xi}+(A_8^4+B_8^4\xi)e^{-8\xi}+A_{10}^4e^{-10\xi}+A_{12}^4e^{-12\xi}]\sin 4\eta\\ &+[-C_1(e^{2\xi}+e^{-2\xi}-2e^{-4\xi})+C_2(e^{2\xi}+e^{-2\xi}+2e^{-4\xi})]\xi\sum_{n=3}^{\infty}e^{-2n\xi}\sin 2n\eta\\ &+[2C_1(e^{2\xi}-1+e^{-2\xi}+e^{-4\xi})-2C_2(e^{2\xi}+1-e^{-2\xi}-e^{-4\xi})]\xi\sum_{n=3}^{\infty}ne^{-2n\xi}\sin 2n\eta\\ &+[-D_1(e^{2\xi}-2-4e^{-2\xi}+6e^{-4\xi}+3e^{-6\xi}-4e^{-8\xi})+D_2(4e^{4\xi}-7e^{2\xi}+2+6e^{-4\xi}-9e^{-6\xi}+4e^{-8\xi})\\ &+D_3(e^{2\xi}+2-4e^{-2\xi}-6e^{-4\xi}+3e^{-6\xi}-4e^{-8\xi})-D_4(4e^{4\xi}-3e^{2\xi}-2+2e^{-4\xi}+3e^{-6\xi}-4e^{-8\xi})]\sum_{n=3}^{\infty}e^{-2n\xi}\sin 2n\eta\\ &+[-2D_1(e^{2\xi}-1-2e^{-2\xi}+2e^{-4\xi}+e^{-6\xi}-e^{-8\xi})-2D_2(2e^{4\xi}-5e^{2\xi}+3+2e^{-2\xi}-4e^{-4\xi}+3e^{-6\xi}-e^{-8\xi})\\ &+2D_3(e^{2\xi}+1-2e^{-2\xi}-2e^{-4\xi}+e^{-6\xi}+e^{-8\xi})+2D_4(2e^{4\xi}-e^{2\xi}-3+2e^{-2\xi}-e^{-6\xi}+e^{-8\xi})]\sum_{n=3}^{\infty}ne^{-2n\xi}\sin 2n\eta\end{aligned}\tag{D7}$$

where

$A_2^2=-96\beta D-8\beta D^2(\gamma+1)(5D+9)-2\beta D^3(\gamma+1)^2(7D+8)$

$A_4^2=80\beta D(D+1)+8\beta D^2(\gamma+1)(6D+1)+\beta D^4(\gamma+1)^2$

$A_6^2=-192\beta D^2-32\beta D^3(\gamma+1)+12\beta D^4(\gamma+1)^2$

$A_8^2=-60\beta D^3(\gamma+1)-15\beta D^4(\gamma+1)^2$

$A_{10}^2=96\beta D^2+48\beta D^3(\gamma+1)+6\beta D^4(\gamma+1)^2$

$A_2^4=\beta D^3(\gamma+1)^2(15D+8)+4\beta D^2(\gamma+1)(7D+9)$

$A_4^4=96\beta D^2-16\beta D^3(\gamma+1)-10\beta D^4(\gamma+1)^2$

$A_6^4=48\beta D^3(\gamma+1)-8\beta D^4(\gamma+1)^2$

$A_8^4=-224\beta D^2-16\beta D^3(\gamma+1)+22\beta D^4(\gamma+1)^2$

$A_{10}^4=-84\beta D^3(\gamma+1)-21\beta D^4(\gamma+1)^2$

$A_{12}^4=128\beta D^2+64\beta D^3(\gamma+1)+8\beta D^4(\gamma+1)^2$

$B_2^2=-8\beta D^2[(\gamma+1)D+4]^2$

$B_4^2=-12\beta D^3(\gamma+1)[(\gamma+1)D+4]$

$B_6^2=16\beta D^2[(\gamma+1)D+4]^2$

$B_2^4=12\beta D^3(\gamma+1)[(\gamma+1)D+4]$

$B_4^4=-16\beta D^2[(\gamma+1)D+4)^2$

$B_6^4=-20\beta D^3(\gamma+1)[(\gamma+1)D+4]$

$B_8^4=24\beta D^2[(\gamma+1)D+4]^2$

$C_1=4\beta D^2[(\gamma+1)D+2][(\gamma+1)D+4]$

$C_2=8\beta D^2[(\gamma+1)D+4]$

$D_1=4\beta D^2[(\gamma+1)D+2]$

$D_2=\beta D^3(\gamma+1)[(\gamma+1)D+2]$

$D_3=8\beta D^2$

$D_4=2\beta D^3(\gamma+1)$

Note that

$$B_6^2=-2B_2^2=-B_4^4=\frac{2}{3}B_8^4=2A_{12}^4=\frac{1}{6}A_{10}^2$$

and

$$B_4^2=-B_2^4=\frac{3}{4}B_6^4=\frac{4}{7}A_{10}^4=\frac{4}{5}A_8^2$$

The right-hand side of equation (D7) suggests a solution of the form

$$\phi_3=G_1(\xi)\sin 2\eta+G_2(\xi)\sin 4\eta+\sum_{n=3}^{\infty}G_n(\xi)\sin 2n\eta\tag{D8}$$

When this expression for ϕ_3 is inserted in the left-hand side of equation (D7) and the coefficients of $\sin 2\eta$, $\sin 4\eta$, and $\sin 2n\eta$ are equated to zero, the following differential equations for $G_1(\xi)$, $G_2(\xi)$, and $G_n(\xi)$ result:

$$\begin{aligned}\frac{d^2G_1}{d\xi^2}-4G_1=&(A_2^2+B_2^2\xi)e^{-2\xi}+(A_4^2+B_4^2\xi)e^{-4\xi}\\ &+(A_6^2+B_6^2\xi)e^{-6\xi}+A_8^2e^{-8\xi}+A_{10}^2e^{-10\xi}\end{aligned}\tag{D9}$$

$$\frac{d^2G_2}{d\xi^2}-16G_2=(A_2^4+B_2^4\xi)e^{-2\xi}+(A_4^4+B_4^4\xi)e^{-4\xi}+(A_6^4+B_6^4\xi)e^{-6\xi}+(A_8^4+B_8^4\xi)e^{-8\xi}+A_{10}^4e^{-10\xi}+A_{12}^4e^{-12\xi}\tag{D10}$$

$$\begin{aligned}\frac{d^2G_n}{d\xi^2}-(2n)^2G_n=&\{4(D_2-D_4)e^{4\xi}+[-D_1-7D_2+D_3+3D_4+(-C_1+C_2)\xi]e^{2\xi}+2(D_1+D_2+D_3+D_4)+[4(D_1-D_3)+(-C_1+C_2)\xi]e^{-2\xi}\\ &+[-6D_1+6D_2-6D_3-2D_4+2(C_1+C_2)\xi]e^{-4\xi}+(-3D_1-9D_2+3D_3-3D_4)e^{-6\xi}\\ &+4(D_1+D_2+D_3+D_4)e^{-8\xi}\}e^{-2n\xi}+\{-4(D_2-D_4)e^{4\xi}+[-2D_1+10D_2+2D_3-2D_4+2(C_1-C_2)\xi]e^{2\xi}\\ &+[2D_1-6D_2+2D_3-6D_4-2(C_1+C_2)\xi]+[4D_1-4D_2-4D_3+4D_4+2(-C_1+C_2)\xi]e^{-2\xi}\\ &+[-4D_1+8D_2-4D_3+2(C_1+C_2)\xi]e^{-4\xi}+(-2D_1-6D_2+2D_3-2D_4)e^{-6\xi}+(2D_1+2D_2+2D_3+2D_4)e^{-8\xi}\}ne^{-2n\xi}\end{aligned}\tag{D11}$$

The solutions of equations (D9), (D10), and (D11) are as follows:

$$G_1(\xi)=-\frac{1}{8}B_2^2\xi^2e^{-2\xi}-\left(\frac{1}{4}A_2^2+\frac{1}{16}B_2^2\right)\xi e^{-2\xi}+\frac{1}{12}B_4^2\xi e^{-4\xi}+\frac{1}{32}B_6^2\xi e^{-6\xi}+\left(\frac{1}{12}A_4^2+\frac{1}{18}B_4^2\right)e^{-4\xi}$$

$$+\left(\frac{1}{32}A_6^2+\frac{3}{256}B_6^2\right)e^{-6\xi}+\frac{1}{60}A_8^2e^{-8\xi}+\frac{1}{96}A_{10}^2e^{-10\xi}+k_1e^{-2\xi} \qquad \text{(D12)}$$

$$G_2(\xi)=-\frac{1}{12}B_2^4\xi e^{-2\xi}+\left(-\frac{1}{12}A_2^4+\frac{1}{36}B_2^4\right)e^{-2\xi}-\frac{1}{16}B_4^4\xi^2e^{-4\xi}-\left(\frac{1}{8}A_4^4+\frac{1}{64}B_4^4\right)\xi e^{-4\xi}+\frac{1}{20}B_6^4\xi e^{-6\xi}+\left(\frac{1}{20}A_6^4+\frac{3}{100}B_6^4\right)e^{-6\xi}$$

$$+\frac{1}{48}B_8^4\xi e^{-8\xi}+\left(\frac{1}{48}A_8^4+\frac{1}{144}B_8^4\right)e^{-8\xi}+\frac{1}{84}A_{10}^4e^{-10\xi}+\frac{1}{128}A_{12}^4e^{-12\xi}+k_2e^{-4\xi} \qquad \text{(D13)}$$

and

$$G_n(\xi)=\frac{1}{4}\beta D^4(\gamma+1)^2e^{-2n\xi+4\xi}-[\beta D^4(\gamma+1)^2+4\beta D^3(\gamma+1)]\xi e^{-2n\xi+2\xi}+\left[-\frac{3}{4}\beta D^4(\gamma+1)^2+\beta D^3(\gamma+1)\right]e^{-2n\xi+2\xi}$$

$$+\frac{1}{2}[3\beta D^4(\gamma+1)^2+8\beta D^3(\gamma+1)-16\beta D^2]\xi e^{-2n\xi}+[\beta D^4(\gamma+1)^2+8\beta D^3(\gamma+1)+16\beta D^2]\xi^2e^{-2n\xi}-\beta D^4(\gamma+1)^2e^{-2n\xi-2\xi}$$

$$-[\beta D^4(\gamma+1)^2+4\beta D^3(\gamma+1)]\xi e^{-2n\xi-2\xi}+\left[\frac{5}{8}\beta D^4(\gamma+1)^2+\beta D^3(\gamma+1)-2\beta D^2\right]e^{-2n\xi-4\xi}$$

$$+\frac{1}{2}[\beta D^4(\gamma+1)^2+8\beta D^3(\gamma+1)+16\beta D^2]\xi e^{-2n\xi-4\xi}-\frac{1}{4}[\beta D^4(\gamma+1)^2+4\beta D^3(\gamma+1)]e^{-2n\xi-6\xi}$$

$$+\frac{1}{16}[\beta D^4(\gamma+1)^2+8\beta D^3(\gamma+1)+16\beta D^2]e^{-2n\xi-8\xi}+k_ne^{-2n\xi} \qquad \text{(D14)}$$

where the following relations have been utilized:

$$C_1-C_2=4\beta D^4(\gamma+1)^2+16\beta D^3(\gamma+1)$$

$$C_1+C_2=4\beta D^4(\gamma+1)^2+32\beta D^3(\gamma+1)+64\beta D^2$$

$$D_1-D_3=4\beta D^3(\gamma+1)$$

$$D_2-D_4=\beta D^4(\gamma+1)^2$$

$$D_1+D_2+D_3+D_4=\beta D^4(\gamma+1)^2+8\beta D^3(\gamma+1)+16\beta D^2=\frac{1}{4}(C_1+C_2)$$

$$D_1+D_2-D_3-D_4=\beta D^4(\gamma+1)^2+4\beta D^3(\gamma+1)=\frac{1}{4}(C_1-C_2)$$

The arbitrary constants k_1, k_2, and $k_n (n\geqq 3)$ are to be determined by the boundary condition at the surface (the boundary condition at infinity is taken care of by putting equal to zero the other set of arbitrary constants that normally appear in the solutions of linear second-order differential equations with constant coefficients). It is now anticipated that the arbitrary constants k_n are independent of n and equal to k, say. Then

$$\sum_{n=3}^{\infty}G_n\sin 2n\eta=G\left[\frac{1}{2}\frac{\sin 2\eta}{\cosh 2\xi-\cos 2\eta}-e^{-2\xi}\sin 2\eta-e^{-4\xi}\sin 4\eta\right] \qquad \text{(D15)}$$

where

$$G=k+\frac{1}{4}\beta D^4(\gamma+1)^2e^{4\xi}+\left[-\frac{3}{4}\beta D^4(\gamma+1)^2+\beta D^3(\gamma+1)\right]e^{2\xi}-\beta D^4(\gamma+1)^2e^{-2\xi}+\left[\frac{5}{8}\beta D^4(\gamma+1)^2+\beta D^3(\gamma+1)-2\beta D^2\right]e^{-4\xi}$$

$$-\frac{1}{4}[\beta D^4(\gamma+1)^2+4\beta D^3(\gamma+1)]e^{-6\xi}+\frac{1}{16}[\beta D^4(\gamma+1)^2+8\beta D^3(\gamma+1)+16\beta D^2]e^{-8\xi}-[\beta D^4(\gamma+1)^2+4\beta D^3(\gamma+1)]\xi e^{2\xi}$$

$$+\frac{1}{2}[3\beta D^4(\gamma+1)^2+8\beta D^3(\gamma+1)-16\beta D^2]\xi+[\beta D^4(\gamma+1)^2+8\beta D^3(\gamma+1)+16\beta D^2]\xi^2-[\beta D^4(\gamma+1)^2+4\beta D^3(\gamma+1)]\xi e^{-2\xi}$$

$$+\frac{1}{2}[\beta D^4(\gamma+1)^2+8\beta D^3(\gamma+1)+16\beta D^2]\xi e^{-4\xi}$$

The expression for the velocity potential ϕ is now

$$\phi=-\cosh\xi\cos\eta-h\phi_1-h^2\phi_2-h^3\phi_3-\ldots \qquad \text{(D16)}$$

where, from equation (B9),

$$\phi_1=\frac{1}{\beta}(e^{-2\xi}\sin 2\eta-2\eta)$$

from equations (C3), (C6), (C7), and (C10),

$$\phi_2=(2D[(\gamma+1)D+4]\xi e^{-\xi}+2De^{-3\xi}+2\{3-D+D[(\gamma+1)D+4]\}e^{-\xi})\cos\eta \qquad \text{(D17)}$$

$$-\left(\frac{1}{2}(\gamma+1)D^2e^{-\xi}+\frac{1}{6}\{12+12D-D[(\gamma+1)D+4]\}e^{-3\xi}\right)\cos 3\eta$$

where

$$D=\frac{1-\beta^2}{\beta^2}$$

and, from equations (D8),

$$\phi_3=G_1(\xi)\sin 2\eta+G_2(\xi)\sin 4\eta+G(\xi)\left(\frac{1}{2}\frac{\sin 2\eta}{\cosh 2\xi-\cos 2\eta}-e^{-2\xi}\sin 2\eta-e^{-4\xi}\sin 4\eta\right)+\Gamma_3\eta \tag{D18}$$

where $G_1(\xi)$ and $G_2(\xi)$ are given by equations (D12) and (D13), respectively, and $G(\xi)$ can be written

$$G(\xi)=\frac{1}{4}Je^{4\xi}-\sqrt{JK}\xi e^{2\xi}+\left(-J+\frac{1}{4}\sqrt{JK}\right)e^{2\xi}-\frac{1}{2}(K-4\sqrt{JK})\xi$$
$$+K\xi^2-Je^{-2\xi}-\sqrt{JK}\xi e^{-2\xi}-\frac{1}{8}(K-2J-4\sqrt{JK})e^{-4\xi}$$
$$+\frac{1}{2}K\xi e^{-4\xi}-\frac{1}{4}\sqrt{JK}e^{-6\xi}+\frac{1}{16}Ke^{-8\xi}+k \tag{D19}$$

with

$$J=\beta D^4(\gamma+1)^2$$
$$K=\beta D^2[(\gamma+1)D+4]^2$$
$$\sqrt{JK}=\beta D^3(\gamma+1)[(\gamma+1)D+4]$$

The arbitrary constants k_1, k_2, and k appearing in the expressions for G_1, G_2, and G, respectively, are determined by the boundary condition at the surface of the circular arc profile. The value of the arbitrary circulation Γ_3 is determined by the Kutta condition at the trailing edge—that the velocity there be finite.

In order to evaluate the various terms appearing in the boundary condition, equation (C9), the following relations are necessary: from equations (A11) and (A12)

$$y=\beta(2h\sin^2\vartheta+8h^3\sin^2\vartheta\cos^2\vartheta)+\ldots$$
$$\frac{dy}{dx}=-4\beta h\cos\vartheta-16\beta h^3\cos\vartheta\cos 2\vartheta-\ldots$$

From equation (B12)

$$b=\sin^2\vartheta-\beta^2(4h^2\sin^4\vartheta+32h^4\sin^4\vartheta\cos^2\vartheta)+\ldots$$
$$\sinh\xi=2\beta h\sin\vartheta\,[1+4h^2\cos^2\vartheta-\beta^2h^2(2\cos^2\vartheta+\sin^4\vartheta)]+\ldots$$
$$\cosh\xi=1+2\beta^2h^2\sin^2\vartheta+\ldots$$
$$e^{\xi}=1+2\beta h\sin\vartheta+2\beta^2h^2\sin^2\vartheta+2\beta h^3\sin\vartheta\,[4\cos^2\vartheta-\beta^2(2\cos^2\vartheta+\sin^4\vartheta)]+\ldots$$
$$e^{-\xi}=1-2\beta h\sin\vartheta+2\beta^2h^2\sin^2\vartheta-2\beta h^3\sin\vartheta\,[4\cos^2\vartheta-\beta^2(2\cos^2\vartheta+\sin^4\vartheta)]+\ldots$$
$$\xi=2\beta h\sin\vartheta+2\beta h^3\sin\vartheta\left[4\cos^2\vartheta-\beta^2\left(\frac{2}{3}\sin^2\vartheta+2\cos^2\vartheta+\sin^4\vartheta\right)\right]+\ldots$$
$$\sin\eta=\sin\vartheta\,(1+2\beta^2h^2\cos^2\vartheta)+\ldots$$
$$\cos\eta=\cos\vartheta\,(1-2\beta^2h^2\sin^2\vartheta)+\ldots$$

When the expression for ϕ given by equation (D16) is substituted into the boundary condition, equation (C9), the coefficient of h^3 on the left-hand side is

$$\{\beta D[(\gamma+1)D+4]-4\beta+6\beta^3\}\cos\vartheta+\{-2\beta D[(\gamma+1)D+4]+2\beta-5\beta^3\}\cos 3\vartheta+\{\beta D[(\gamma+1)D+4]+2\beta-\beta^3\}\cos 5\vartheta \tag{D20}$$

and the coefficient of h^3 on the right-hand side is

$$\beta^3\left[-6-\frac{20}{3}D+\frac{7}{3}D^2-\frac{17}{3}D^2(\gamma+1)-\frac{35}{6}D^3(\gamma+1)-2D^3(\gamma+1)^2-\frac{53}{48}D^4(\gamma+1)^2+\frac{k_1-k}{\beta}\right]\cos\vartheta$$
$$+\beta^3\left[13+\frac{32}{3}D-\frac{7}{3}D^2+\frac{5}{3}D^2(\gamma+1)+\frac{29}{6}D^3(\gamma+1)+\frac{4}{3}D^3(\gamma+1)^2+\frac{15}{16}D^4(\gamma+1)^2+\left(\frac{-k_1+2k_2-k}{\beta}\right)\right]\cos 3\vartheta$$
$$+\beta^3\left[-7-4D+4D^2(\gamma+1)+D^3(\gamma+1)+\frac{2}{3}D^3(\gamma+1)^2+\frac{1}{6}D^4(\gamma+1)^2+\left(\frac{-2k_2+2k}{\beta}\right)\right]\cos 5\vartheta \tag{D21}$$

In obtaining expression (D21), the following relations were utilized:

$$\left.\begin{aligned}
G_1(0)&=\frac{20}{3}\beta D+\frac{14}{3}\beta D^2+\frac{2}{3}\beta D^2(\gamma+1)+\frac{4}{3}\beta D^3(\gamma+1)-\frac{5}{24}\beta D^4(\gamma+1)^2+k_1\\
G_2(0)&=-\beta D^2-32\beta D^2(\gamma+1)-\frac{1}{2}\beta D^3(\gamma+1)-\frac{2}{3}\beta D^3(\gamma+1)^2-\frac{71}{48}\beta D^4(\gamma+1)^2+k_2\\
G(0)&=-\frac{17}{16}\beta D^4(\gamma+1)^2+\frac{3}{2}\beta D^3(\gamma+1)-\beta D^2+k\\
\left(\frac{dG_1}{d\xi}\right)_0&=-\frac{8}{3}\beta D-\frac{8}{3}\beta D^2+\frac{46}{3}\beta D^2(\gamma+1)+\frac{26}{3}\beta D^3(\gamma+1)+4\beta D^3(\gamma+1)^2+\frac{13}{3}\beta D^4(\gamma+1)^2-2k_1\\
\left(\frac{dG_2}{d\xi}\right)_0&=4\beta D^2+6\beta D^2(\gamma+1)-4\beta D^3(\gamma+1)+\frac{4}{3}\beta D^3(\gamma+1)^2+\frac{55}{12}\beta D^4(\gamma+1)^2-4k_2\\
\left(\frac{dG}{d\xi}\right)_0&=0
\end{aligned}\right\} \tag{D22}$$

By equating the coefficients of $\cos\vartheta$, $\cos 3\vartheta$, and $\cos 5\vartheta$ appearing in expressions (D20) and (D21), the following equations for the arbitrary constants $k_1, k_2,$ and k are obtained:

$$\frac{1}{\beta}(k_1-k)=\frac{D}{\beta^2}[(\gamma+1)D+4]-\frac{4}{\beta^2}+12+\frac{20}{3}D-\frac{7}{3}D^2+\frac{17}{3}D^2(\gamma+1)+\frac{35}{6}D^3(\gamma+1)+2D^3(\gamma+1)^2+\frac{53}{48}D^4(\gamma+1)^2 \tag{D23}$$

$$\frac{1}{\beta}(-k_1+2k_2-k)=-\frac{2D}{\beta^2}[(\gamma+1)D+4]+\frac{2}{\beta^2}-18-\frac{32}{3}D+\frac{7}{3}D^2-\frac{5}{3}D^2(\gamma+1)-\frac{29}{6}D^3(\gamma+1)-\frac{4}{3}D^3(\gamma+1)^2-\frac{15}{16}D^4(\gamma+1)^2 \tag{D24}$$

$$\frac{1}{\beta}(-2k_2+2k)=\frac{D}{\beta^2}[(\gamma+1)D+4]+\frac{2}{\beta^2}+6+4D-4D^2(\gamma+1)-D^3(\gamma+1)-\frac{2}{3}D^3(\gamma+1)^2-\frac{1}{6}D^4(\gamma+1)^2 \tag{D25}$$

Note that the sum of equations (D24) and (D25) yields equation (D23), so that these equations for k_1, k_2, and k are not independent. Hence, one of the constants, say k, is entirely arbitrary. It will be seen in the following discussion that the arbitrary disposal of k is necessary in order that no infinite velocities occur on the circular arc profile.

The velocity components along the X- and Y-axes are given by

$$\left.\begin{aligned} u&=\frac{\partial\phi}{\partial X}=\frac{2}{\cosh 2\xi-\cos 2\eta}\left(\sinh\xi\cos\eta\frac{\partial\phi}{\partial\xi}-\cosh\xi\sin\eta\frac{\partial\phi}{\partial\eta}\right)\\ v&=\frac{\partial\phi}{\partial Y}=\frac{2\beta}{\cosh 2\xi-\cos 2\eta}\left(\cosh\xi\sin\eta\frac{\partial\phi}{\partial\xi}+\sinh\xi\cos\eta\frac{\partial\phi}{\partial\eta}\right)\end{aligned}\right\} \tag{D26}$$

Along the chord of the circular arc profile, $\xi=0$; equations (D26) therefore become

$$\left.\begin{aligned} u&=-\frac{1}{\sin\eta}\frac{\partial\phi}{\partial\eta}\\ v&=\frac{\beta}{\sin\eta}\frac{\partial\phi}{\partial\xi}\end{aligned}\right\} \tag{D27}$$

By means of equation (D16) for ϕ and the expressions for ϕ_1, ϕ_2, and ϕ_3, it follows easily from equations (D27) that

$$\begin{aligned} u=&-1-\frac{4h}{\beta}\sin\eta+h^2\{12\cos 2\eta+D[(\gamma+1)D+4](2\cos 2\eta-1)\}\\ &+\frac{h^3}{\sin\eta}[2G_1(0)\cos 2\eta+4G_2(0)\cos 4\eta+\Gamma_3]+\frac{h^3}{\sin\eta}G(0)\left[-2\cos 2\eta-4\cos 4\eta-\cos^2\eta+\frac{\cos 2\eta}{2\sin^2\eta}\right]+\cdots\end{aligned} \tag{D28}$$

$$\begin{aligned} v=&4h\cos\eta+4\beta h^2(2D+3)\sin 2\eta\\ &-2\beta h^3\cos\eta\left[\left(\frac{dG_1}{d\xi}\right)_0+2\left(\frac{dG_2}{d\xi}\right)_0+\left(\frac{dG}{d\xi}\right)_0\left(\frac{1}{4\sin^2\eta}-1-2\cos 2\eta\right)+2G(0)(1+4\cos 2\eta)\right]+\cdots\end{aligned} \tag{D29}$$

At the trailing edge, $\eta=\pi$ or $\sin\eta=0$. Hence, according to equations (D28) and (D29), an infinite velocity seems to occur there. The Kutta condition at the trailing edge, however, demands that the velocity be finite. From equations (D22) it is seen that $\left(\frac{dG}{d\xi}\right)_0=0$ so that the velocity component v is finite on the boundary. The velocity component u can be rendered finite by showing that the coefficients of $\frac{h^3}{\sin\eta}$ in equation (D28) can be made to equal zero when $\eta=0$ or π. Thus, since the constant k occurring in equation (D15) is arbitrary, it can be chosen so that $G(0)=0$. Again, if the first coefficient of $\frac{h^3}{\sin\eta}$ in equation (D28) vanishes for $\eta=\pi$, then the circulation constant

$$\Gamma_3=-2G_1(0)-4G_2(0) \tag{D30}$$

where $G_1(0)$ and $G_2(0)$ are given by equations (D22).

The arbitrary constant k has been determined by the condition $G(0)=0$. From equations (D22), therefore,

$$\frac{k}{\beta}=\frac{17}{16}D^4(\gamma+1)^2-\frac{3}{2}D^3(\gamma+1)+D^2 \tag{D31}$$

and from equations (D23) and (D25), respectively,

$$\begin{aligned}\frac{k_1}{\beta}=&\frac{D}{\beta^2}[(\gamma+1)D+4]-\frac{4}{\beta^2}+12+\frac{20}{3}D-\frac{4}{3}D^2+\frac{17}{3}D^2(\gamma+1)\\ &+\frac{13}{3}D^3(\gamma+1)+2D^3(\gamma+1)^2+\frac{13}{6}D^4(\gamma+1)^2\end{aligned} \tag{D32}$$

$$\begin{aligned}\frac{k_2}{\beta}=&-\frac{D}{2\beta^2}[(\gamma+1)D+4]-\frac{1}{\beta^2}-3-2D+D^2+2D^2(\gamma+1)\\ &-D^3(\gamma+1)+\frac{1}{3}D^3(\gamma+1)^2+\frac{55}{48}D^4(\gamma+1)^2\end{aligned} \tag{D33}$$

Note that, had the incompressible flow past a circular arc profile been determined according to the methods of the present paper, a discussion similar to the foregoing would have been necessary, with the result that $k_1=8$, $k_2=-4$, $k=0$, and $\Gamma_3=0$.

Substituting from equations (D21) for $G_1(0)$ and $G_2(0)$ into equation (D30) gives

$$\Gamma_3=-\frac{20}{3}\,\beta D-\frac{20}{3}\,\beta D^2-\frac{26}{3}\,\beta D^2\,(\gamma+1)-\frac{16}{3}\,\beta D^3\,(\gamma+1)$$
$$-\frac{31}{12}\,\beta D^4\,(\gamma+1)^2-\frac{8}{3}\,\beta D^3\,(\gamma+1)^2 \tag{D34}$$

The circulation Γ_i in the incompressible case, obtained from equation (A5), is

$$\frac{\Gamma_i}{4\pi Ua}=2h$$

The circulation Γ_c in the compressible case, inclusive of terms containing the third power of h, is obtained by adding the circulation term from equation (B9) to the value of Γ_3 given by equation (D34) and multiplying the result by $4\pi Ua$. Thus, if D is replaced by $\frac{1-\beta^2}{\beta^2}$,

$$\frac{\Gamma_c}{4\pi Ua}=\frac{2}{\beta}\,h+\left[\frac{20}{3}\,\frac{1-\beta^2}{\beta^3}+\frac{2}{3}\,(\gamma+1)\,\frac{(1-\beta^2)^2}{\beta^5}\,(8+5\beta^2)\right.$$
$$\left.+\frac{1}{12}\,(\gamma+1)^2\,\frac{(1-\beta^2)^3}{\beta^7}\,(31+\beta^2)\right]h^3 \tag{D35}$$

The circulation correction formula then becomes

$$\frac{\Gamma_c}{\Gamma_i}=\frac{1}{\beta}+\left[\frac{10}{3}\,\frac{1-\beta^2}{\beta^3}+\frac{1}{3}\,(\gamma+1)\,\frac{(1-\beta^2)^2}{\beta^5}\,(8+5\beta^2)\right.$$
$$\left.+\frac{1}{24}\,(\gamma+1)^2\,\frac{(1-\beta^2)^3}{\beta^7}\,(31+\beta^2)\right]h^2 \tag{D36}$$

The first term on the right-hand side is the familiar Prandtl-Glauert term so that the second term represents the first departure from the Prandtl-Glauert rule.

The magnitude of the velocity at the surface of the circular arc profile is calculated by the use of equations (D26). Thus

$$q=1+h\,\frac{\partial\phi_1}{\partial x}+h^2\left[\frac{\beta^2}{2}\left(\frac{\partial\phi_1}{\partial y}\right)^2+\frac{\partial\phi_2}{\partial x}\right]$$
$$+h^3\left[-\frac{\beta^2}{2}\,\frac{\partial\phi_1}{\partial x}\left(\frac{\partial\phi_1}{\partial y}\right)^2+\beta^2\,\frac{\partial\phi_1}{\partial y}\,\frac{\partial\phi_2}{\partial y}+\frac{\partial\phi_3}{\partial x}\right]+\;.\;.\;. \tag{D37}$$

where, symbolically,

$$\frac{\partial}{\partial x}=\frac{2}{\cosh 2\xi-\cos 2\eta}\left(\sinh\xi\,\cos\eta\,\frac{\partial}{\partial\xi}-\cosh\xi\,\sin\eta\,\frac{\partial}{\partial\eta}\right)$$
$$\frac{\partial}{\partial y}=\frac{2}{\cosh 2\xi-\cos 2\eta}\left(\cosh\xi\,\sin\eta\,\frac{\partial}{\partial\xi}+\sinh\xi\,\cos\eta\,\frac{\partial}{\partial\eta}\right)$$

and the expressions for ϕ_1, ϕ_2, and ϕ_3 are given by equations (B9), (D17), and (D18). When all the functions of ξ and η are expressed as functions of ϑ at the surface of the profile and terms involving powers of h higher than the third are neglected, the expression for q becomes

$$q=1+\frac{4h}{\beta}\sin\vartheta+h^2\left\{-2-\frac{2}{\beta^4}-(\gamma-1)\left(\frac{1-\beta^2}{\beta^2}\right)^2+4\left[\frac{2}{\beta^4}+(\gamma-1)\left(\frac{1-\beta^2}{\beta^2}\right)^2\right]\sin^2\vartheta\right\}$$
$$+h^3\left\{4\left[-\frac{2}{\beta}+G_1(0)+2G_2(0)\right]\sin\vartheta+8\left[-\frac{2}{\beta}+2\beta(2D+3)+G_2(0)\right]\sin 3\vartheta\right\}+\;.\;.\;. \tag{D38}$$

where, from equations (D22) and (D32),

$$G_1(0)=-\frac{4}{\beta}+12\beta+\frac{4}{3}\,D\left(\frac{3}{\beta}+10\beta\right)+\frac{10}{3}\,\beta D^2+\frac{1}{3}\,D^2(\gamma+1)\left(\frac{3}{\beta}+19\beta+17\beta D\right)+\frac{1}{24}\,\beta D^3\,(\gamma+1)^2\,(48+47D)$$

and, from equations (D22) and (D33),

$$G_2(0)=-\frac{1}{\beta}-3\beta-2D\left(\frac{1}{\beta}+\beta\right)-\frac{1}{2}\,D^2\,(\gamma+1)\left(\frac{1}{\beta}+2\beta+3\beta D\right)-\frac{1}{3}\,\beta D^3\,(\gamma+1)^2\,(1+D)$$

If q_c and q_i denote the magnitude of the velocity at the boundary in the compressible and the incompressible cases, respectively, then the velocity correction formula is

$$\frac{q_c}{q_i}=\frac{q}{1+4h\sin\vartheta-4h^2\cos 2\vartheta-8h^3\sin\vartheta} \tag{D39}$$

where q is obtained from equation (D38) and where $0\leqq\vartheta\leqq\pi$ for the upper side of the circular arc and $-\pi\leqq\vartheta\leqq 0$ for the lower side of the circular arc. For the leading or trailing edge, $\vartheta=0$ or $\vartheta=\pi$, the velocity ratio q_c/q_i is again given by equation (C16). For the position of maximum velocity, $\vartheta=\frac{\pi}{2}$,

$$\frac{q_c}{q_i}=\frac{1+\frac{4h}{\beta}+h^2\left[-8+3\left(\frac{1}{\beta^2}+1\right)^2+3\gamma\left(\frac{1}{\beta^2}-1\right)^2\right]+4h^3[-2\beta(4D+5)+G_1(0)]}{1+4h+4h^2-8h^3} \tag{D40}$$

For the position of minimum velocity, $\vartheta=-\frac{\pi}{2}$,

$$\frac{q_c}{q_i}=\frac{1-\frac{4h}{\beta}+h^2\left[-8+3\left(\frac{1}{\beta^2}+1\right)^2+3\gamma\left(\frac{1}{\beta^2}-1\right)^2\right]-4h^3[-2\beta(4D+5)+G_1(0)]}{1-4h+4h^2+8h^3} \tag{D41}$$

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TABLE I.—RATIO OF CIRCULATIONS FOR COMPRESSIBLE AND INCOMPRESSIBLE FLOWS

Approximation \ M_1	Γ_c/Γ_i													
	0.10	0.20	0.30	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90
Prandtl-Glauert	1.0050	1.0206	1.0483	1.0911	1.1198	1.1547	1.1974	1.2500	1.3159	1.4003	1.5119	1.6667	1.8983	2.2942
h=0.010														
Third	1.0050	1.0206	1.0483	1.0912	1.1200	1.1550	1.1978	1.2508	1.3172	1.4029	1.5170	1.6788	1.9348	2.4603
von Kármán	1.0050	1.0206	1.0483	1.0911	1.1199	1.1548	1.1975	1.2503	1.3165	1.4013	1.5135	1.6692	1.9025	2.3030
h=0.015														
Third	1.0051	1.0207	1.0484	1.0913	1.1202	1.1553	1.1984	1.2517	1.3189	1.4059	1.5234	1.6940	1.9804	2.6630
von Kármán	1.0050	1.0207	1.0484	1.0913	1.1202	1.1553	1.1983	1.2511	1.3175	1.4026	1.5152	1.6720	1.9078	2.3142
h=0.020														
Third	1.0051	1.0207	1.0485	1.0915	1.1205	1.1558	1.1992	1.2530	1.3212	1.4102	1.5323	1.7153	2.0441	2.9588
von Kármán	1.0051	1.0207	1.0485	1.0915	1.1205	1.1557	1.1987	1.2520	1.3187	1.4043	1.5179	1.6763	1.9153	2.3301
h=0.025														
Third	1.0051	1.0207	1.0486	1.0918	1.1209	1.1565	1.2002	1.2547	1.3242	1.4158	1.5438	1.7427	2.1262	------
von Kármán	1.0051	1.0207	1.0485	1.0916	1.1206	1.1560	1.1996	1.2531	1.3203	1.4066	1.5213	1.6818	1.9250	------
h=0.030														
Third	1.0051	1.0208	1.0487	1.0921	1.1214	1.1572	1.2015	1.2568	1.3278	1.4226	1.5578	1.7762	2.2264	------
von Kármán	1.0051	1.0208	1.0486	1.0920	1.1212	1.1570	1.2006	1.2544	1.3222	1.4094	1.5256	1.6885	1.9370	------
h=0.035														
Third	1.0051	1.0208	1.0488	1.0925	1.1220	1.1581	1.2030	1.2593	1.3321	1.4307	1.5744	1.8157	2.3449	------
von Kármán	1.0051	1.0208	1.0486	1.0922	1.1215	1.1575	1.2017	1.2560	1.3245	1.4127	1.5306	1.6966	1.9514	------
h=0.040														
Third	1.0051	1.0209	1.0490	1.0929	1.1226	1.1592	1.2047	1.2621	1.3371	1.4400	1.5936	1.8813	------	------
von Kármán	1.0051	1.0209	1.0488	1.0926	1.1220	1.1588	1.2031	1.2579	1.3271	1.4166	1.5364	1.7060	------	------
h=0.045														
Third	1.0051	1.0210	1.0492	1.0934	1.1234	1.1604	1.2066	1.2653	1.3427	1.4505	1.6153	1.9130	------	------
von Kármán	1.0051	1.0210	1.0490	1.0930	1.1228	1.1599	1.2046	1.2600	1.3301	1.4209	1.5430	1.7168	------	------
h=0.050														
Third	1.0051	1.0210	1.0494	1.0939	1.1242	1.1617	1.2087	1.2689	1.3490	1.4623	1.6396	1.9708	------	------
von Kármán	1.0051	1.0210	1.0492	1.0936	1.1238	1.1611	1.2063	1.2624	1.3335	1.4258	1.5505	1.7290	------	------
h=0.060														
Third	1.0052	1.0212	1.0499	1.0952	1.1262	1.1648	1.2137	1.2773	1.3636	1.4895	1.6958	------	------	------
von Kármán	1.0052	1.0210	1.0496	1.0950	1.1260	1.1640	1.2102	1.2679	1.3413	1.4373	1.5681	------	------	------
h=0.070														
Third	1.0052	1.0214	1.0505	1.0967	1.1285	1.1685	1.2196	1.2871	1.3808	1.5218	1.7622	------	------	------
von Kármán	1.0052	1.0212	1.0500	1.0960	1.1281	1.1673	1.2148	1.2744	1.3507	1.4511	1.5895	------	------	------
h=0.080														
Third	1.0053	1.0217	1.0512	1.0984	1.1312	1.1727	1.2265	1.2985	1.4007	1.5589	------	------	------	------
von Kármán	1.0052	1.0215	1.0510	1.0976	1.1289	1.1711	1.2202	1.2820	1.3616	1.4673	------	------	------	------
h=0.090														
Third	1.0053	1.0219	1.0520	1.1003	1.1342	1.1775	1.2342	1.3144	1.4232	1.6011	------	------	------	------
von Kármán	1.0052	1.0217	1.0515	1.1001	1.1340	1.1755	1.2263	1.2907	1.3741	1.4861	------	------	------	------
h=0.100														
Third	1.0054	1.0222	1.0528	1.1025	1.1376	1.1828	1.2428	1.3258	1.4484	1.6482	------	------	------	------
von Kármán	1.0053	1.0220	1.0526	1.1020	1.1370	1.1804	1.2332	1.3004	1.3883	1.5076	------	------	------	------

TABLE II.—RATIO OF VELOCITIES AT LEADING OR TRAILING EDGE FOR COMPRESSIBLE AND INCOMPRESSIBLE FLOWS

	q_c/q_i														
Approximation \ M_1	0.10	0.20	0.30	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
$h=0.010;\ (q_i)_{exact}=\frac{1}{1.0004}$															
First	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9999	0.9999	0.9999	0.9998	0.9998	0.9997	0.9996	0.9995	0.9991
Third	1.0000	1.0000	1.0000	.9999	.9999	.9998	.9998	.9997	.9996	.9994	.9991	.9985	.9973	.9939	.9757
$h=0.015;\ (q_i)_{exact}=\frac{1}{1.0009}$															
First	1.0000	1.0000	1.0000	0.9999	0.9999	0.9999	0.9998	0.9998	0.9997	0.9996	0.9995	0.9994	0.9992	0.9988	0.9980
Third	1.0000	1.0000	.9999	.9998	.9997	.9996	.9995	.9993	.9991	.9986	.9980	.9967	.9940	.9803	.9154
$h=0.020;\ (q_i)_{exact}=\frac{1}{1.0016}$															
First	1.0000	1.0000	0.9999	0.9999	0.9998	0.9998	0.9997	0.9996	0.9995	0.9994	0.9992	0.9989	0.9986	0.9979	0.9965
Third	1.0000	.9999	.9998	.9997	.9995	.9994	.9991	.9988	.9983	.9976	.9964	.9941	.9893	.9757	.9028
$h=0.025;\ (q_i)_{exact}=\frac{1}{1.0025}$															
First	1.0000	1.0000	0.9999	0.9998	0.9997	0.9996	0.9995	0.9994	0.9992	0.9990	0.9987	0.9983	0.9978	0.9968	0.9915
Third	1.0000	.9999	.9997	.9995	.9993	.9990	.9986	.9981	.9974	.9962	.9943	.9908	.9833	.9620	.8480
$h=0.030;\ (q_i)_{exact}=\frac{1}{1.0036}$															
First	1.0000	0.9999	0.9998	0.9997	0.9996	0.9994	0.9993	0.9991	0.9989	0.9986	0.9982	0.9976	0.9968	0.9953	0.9921
Third	1.0000	.9999	.9996	.9992	.9990	.9986	.9980	.9973	.9962	.9945	.9918	.9867	.9759	.9452	.7808
$h=0.035;\ (q_i)_{exact}=\frac{1}{1.0049}$															
First	1.0000	0.9999	0.9998	0.9996	0.9994	0.9992	0.9990	0.9988	0.9985	0.9980	0.9975	0.9967	0.9956	0.9937	0.9892
Third	.9999	.9998	.9995	.9990	.9986	.9980	.9973	.9963	.9948	.9925	.9888	.9819	.9672	.9253	.7013
$h=0.040;\ (q_i)_{exact}=\frac{1}{1.0064}$															
First	1.0000	0.9999	0.9997	0.9994	0.9992	0.9990	0.9987	0.9984	0.9980	0.9974	0.9967	0.9957	0.9943	0.9917	0.9859
Third	.9999	.9997	.9993	.9986	.9981	.9974	.9965	.9952	.9932	.9903	.9853	.9763	.9570	.9023	.6092
$h=0.045;\ (q_i)_{exact}=\frac{1}{1.0081}$															
First	1.0000	0.9998	0.9996	0.9993	0.9990	0.9988	0.9984	0.9980	0.9974	0.9968	0.9959	0.9946	0.9927	0.9895	0.9822
Third	.9999	.9997	.9991	.9983	.9976	.9967	.9955	.9939	.9914	.9876	.9814	.9700	.9455	.8761	.5010
$h=0.050;\ (q_i)_{exact}=\frac{1}{1.01}$															
First	1.0000	0.9998	0.9995	0.9991	0.9988	0.9985	0.9980	0.9975	0.9968	0.9960	0.9949	0.9933	0.9910	0.9871	0.9780
Third	.9999	.9996	.9989	.9979	.9970	.9960	.9945	.9924	.9894	.9847	.9770	.9629	.9326	.8468	.3872
$h=0.060;\ (q_i)_{exact}=\frac{1}{1.0144}$															
First	0.9999	0.9997	0.9993	0.9987	0.9983	0.9978	0.9972	0.9964	0.9955	0.9942	0.9926	0.9904	0.9871	0.9814	0.9683
Third	.9999	.9994	.9985	.9969	.9957	.9942	.9920	.9890	.9846	.9779	.9667	.9463	.9025	.7784	.1137
$h=0.070;\ (q_i)_{exact}=\frac{1}{1.0196}$															
First	0.9999	0.9996	0.9991	0.9982	0.9977	0.9970	0.9961	0.9951	0.9938	0.9922	0.9900	0.9869	0.9824	0.9746	0.9568
Third	.9998	.9991	.9979	.9958	.9942	.9920	.9891	.9850	.9790	.9697	.9545	.9266	.8666	.6968	----------
$h=0.080;\ (q_i)_{exact}=\frac{1}{1.0256}$															
First	0.9999	0.9995	0.9988	0.9977	0.9969	0.9960	0.9950	0.9936	0.9919	0.9898	0.9869	0.9829	0.9770	0.9669	0.9430
Third	.9997	.9989	.9973	.9944	.9923	.9895	.9856	.9802	.9723	.9602	.9402	.9035	.8247	.6015	----------
$h=0.090;\ (q_i)_{exact}=\frac{1}{1.0324}$															
First	0.9998	0.9993	0.9984	0.9971	0.9961	0.9950	0.9936	0.9919	0.9898	0.9870	0.9834	0.9784	0.9709	0.9581	0.9280
Third	.9997	.9986	.9965	.9929	.9902	.9866	.9817	.9748	.9648	.9493	.9237	.8770	.7766	.4921	----------
$h=0.100;\ (q_i)_{exact}=\frac{1}{1.04}$															
First	0.9998	0.9992	0.9981	0.9964	0.9952	0.9938	0.9921	0.9900	0.9874	0.9840	0.9795	0.9733	0.9641	0.9482	0.9110
Third	.9996	.9982	.9956	.9912	.9878	.9833	.9772	.9687	.9561	.9369	.9051	.8469	.7221	.3680	----------

TABLE III.—RATIO OF MAXIMUM VELOCITIES FOR COMPRESSIBLE AND INCOMPRESSIBLE FLOWS

Approximation \ M_l	q_{max_c}/q_{max_i}														
	0.10	0.20	0.30	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
$h=0.010$; $(q_i)_1=1.04$; $(q_i)_2=1.0404$; $(q_i)_3=1.0404$; $(q_i)_{exact}=1.0404$															
First	1.0002	1.0008	1.0019	1.0035	1.0046	1.0060	1.0076	1.0096	1.0122	1.0154	1.0197	1.0256	1.0346	1.0498	1.0847
Second	1.0002	1.0008	1.0020	1.0037	1.0049	1.0064	1.0082	1.0105	1.0134	1.0171	1.0223	1.0299	1.0422	1.0673	1.1547
Third	1.0002	1.0008	1.0020	1.0038	1.0050	1.0064	1.0083	1.0105	1.0135	1.0173	1.0227	1.0307	1.0446	1.0779	1.2829
$h=0.015$; $(q_i)_1=1.06$; $(q_i)_2=1.0609$; $(q_i)_3=1.0609$; $(q_i)_{exact}=1.0309$															
First	1.0003	1.0012	1.0027	1.0052	1.0068	1.0088	1.0112	1.0142	1.0179	1.0227	1.0290	1.0377	1.0509	1.0733	1.1247
Second	1.0003	1.0013	1.0030	1.0057	1.0075	1.0098	1.0126	1.0161	1.0205	1.0265	1.0347	1.0471	1.0678	1.1118	1.2790
Third	1.0003	1.0013	1.0030	1.0057	1.0076	1.0098	1.0127	1.0162	1.0209	1.0271	1.0359	1.0498	1.0758	1.1469	1.7034
$h=0.020$; $(q_i)_1=1.08$; $(q_i)_2=1.0816$; $(q_i)_3=1.0815$; $(q_i)_{exact}=1.0815$															
First	1.0004	1.0015	1.0036	1.0068	1.0089	1.0115	1.0146	1.0185	1.0234	1.0297	1.0379	1.0494	1.0665	1.0957	1.1631
Second	1.0004	1.0017	1.0040	1.0077	1.0102	1.0132	1.0170	1.0218	1.0280	1.0363	1.0480	1.0656	1.0961	1.1630	1.4321
Third	1.0004	1.0017	1.0041	1.0077	1.0103	1.0134	1.0173	1.0222	1.0288	1.0377	1.0507	1.0720	1.1146	1.2448	2.4191
$h=0.025$; $(q_i)_1=1.10$; $(q_i)_2=1.1025$; $(q_i)_3=1.1024$; $(q_i)_{exact}=1.1024$															
First	1.0005	1.0019	1.0044	1.0083	1.0109	1.0141	1.0179	1.0227	1.0287	1.0364	1.0465	1.0606	1.0817	1.1177	1.2002
Second	1.0005	1.0022	1.0051	1.0097	1.0129	1.0168	1.0216	1.0278	1.0358	1.0466	1.0619	1.0855	1.1269	1.2206	1.5308
Third	1.0005	1.0022	1.0052	1.0098	1.0130	1.0171	1.0221	1.0286	1.0372	1.0492	1.0671	1.0976	1.1624	1.3772	3.4221
$h=0.030$; $(q_i)_1=1.12$; $(q_i)_2=1.1236$; $(q_i)_3=1.1234$; $(q_i)_{exact}=1.1234$															
First	1.0005	1.0022	1.0052	1.0098	1.0128	1.0166	1.0212	1.0268	1.0339	1.0429	1.0548	1.0714	1.0963	1.1387	1.2380
Second	1.0006	1.0026	1.0062	1.0118	1.0156	1.0204	1.0263	1.0339	1.0439	1.0573	1.0766	1.1065	1.1601	1.2840	1.8183
Third	1.0006	1.0026	1.0063	1.0120	1.0159	1.0209	1.0272	1.0353	1.0462	1.0616	1.0854	1.1271	1.2203	1.5496	5.0253
$h=0.035$; $(q_i)_1=1.14$; $(q_i)_2=1.1449$; $(q_i)_3=1.1446$; $(q_i)_{exact}=1.1446$															
First	1.0006	1.0025	1.0059	1.0112	1.0147	1.0190	1.0242	1.0307	1.0388	1.0492	1.0629	1.0819	1.1103	1.1589	1.2705
Second	1.0007	1.0031	1.0073	1.0139	1.0184	1.0241	1.0312	1.0402	1.0522	1.0684	1.0918	1.1287	1.1955	1.3530	2.0483
Third	1.0008	1.0031	1.0074	1.0142	1.0189	1.0248	1.0324	1.0423	1.0558	1.0752	1.1056	1.1608	1.2894	1.7670	7.0467
$h=0.040$; $(q_i)_1=1.16$; $(q_i)_2=1.1664$; $(q_i)_3=1.1659$; $(q_i)_{exact}=1.1659$															
First	1.0007	1.0028	1.0067	1.0126	1.0165	1.0213	1.0272	1.0345	1.0436	1.0552	1.0706	1.0920	1.1239	1.1785	1.3038
Second	1.0009	1.0035	1.0084	1.0160	1.0213	1.0278	1.0361	1.0467	1.0607	1.0798	1.1077	1.1519	1.2330	1.4272	2.3007
Third	1.0009	1.0036	1.0085	1.0165	1.0220	1.0290	1.0379	1.0498	1.0660	1.0898	1.1279	1.1990	1.3706	2.0339	9.6255
$h=0.045$; $(q_i)_1=1.18$; $(q_i)_2=1.1881$; $(q_i)_3=1.1874$; $(q_i)_{exact}=1.1874$															
First	1.0008	1.0032	1.0074	1.0139	1.0183	1.0236	1.0301	1.0381	1.0482	1.0611	1.0781	1.1017	1.1370	1.1974	1.3360
Second	1.0010	1.0040	1.0095	1.0182	1.0241	1.0316	1.0411	1.0533	1.0694	1.0916	1.1241	1.1762	1.2725	1.5063	2.5745
Third	1.0010	1.0041	1.0097	1.0188	1.0252	1.0332	1.0437	1.0576	1.0769	1.1055	1.1523	1.2420	1.4649	2.3546	12.8152
$h=0.050$; $(q_i)_1=1.20$; $(q_i)_2=1.2100$; $(q_i)_3=1.2090$; $(q_i)_{exact}=1.2089$															
First	1.0008	1.0034	1.0081	1.0152	1.0200	1.0258	1.0329	1.0417	1.0527	1.0667	1.0853	1.1111	1.1497	1.2157	1.3671
Second	1.0011	1.0045	1.0106	1.0203	1.0270	1.0355	1.0462	1.0600	1.0783	1.1037	1.1411	1.2013	1.3139	1.5900	2.8682
Third	1.0011	1.0046	1.0109	1.0212	1.0284	1.0377	1.0497	1.0658	1.0884	1.1224	1.1791	1.2900	1.5731	2.7327	16.6647
$h=0.060$; $(q_i)_1=1.24$; $(q_i)_2=1.2544$; $(q_i)_3=1.2527$; $(q_i)_{exact}=1.2525$															
First	1.0010	1.0040	1.0094	1.0176	1.0232	1.0299	1.0382	1.0484	1.0611	1.0775	1.0991	1.1290	1.1739	1.2505	1.4263
Second	1.0013	1.0054	1.0129	1.0247	1.0330	1.0434	1.0566	1.0737	1.0967	1.1288	1.1764	1.2541	1.4016	1.7700	3.5106
Third	1.0014	1.0056	1.0135	1.0262	1.0353	1.0470	1.0625	1.0835	1.1136	1.1599	1.2398	1.4021	1.8341	3.6761	26.5210
$h=0.070$; $(q_i)_1=1.28$; $(q_i)_2=1.2996$; $(q_i)_3=1.2969$; $(q_i)_{exact}=1.2965$															
First	1.0011	1.0045	1.0106	1.0199	1.0262	1.0338	1.0432	1.0547	1.0691	1.0876	1.1120	1.1458	1.1965	1.2831	1.4818
Second	1.0016	1.0064	1.0151	1.0292	1.0391	1.0514	1.0673	1.0879	1.1157	1.1548	1.2133	1.3099	1.4954	1.9651	4.2193
Third	1.0016	1.0067	1.0161	1.0315	1.0426	1.0570	1.0763	1.1028	1.1416	1.2027	1.3106	1.5371	2.1589	4.8892	39.5150
$h=0.080$; $(q_i)_1=1.32$; $(q_i)_2=1.3456$; $(q_i)_3=1.3415$; $(q_i)_{exact}=1.3409$															
First	1.0012	1.0050	1.0117	1.0221	1.0290	1.0375	1.0479	1.0606	1.0766	1.0970	1.1241	1.1616	1.2178	1.3137	1.5339
Second	1.0018	1.0073	1.0175	1.0338	1.0452	1.0596	1.0781	1.1024	1.1352	1.1816	1.2517	1.3682	1.5944	2.1735	4.9863
Third	1.0019	1.0078	1.0188	1.0371	1.0504	1.0678	1.0912	1.1240	1.1776	1.2508	1.3923	1.6963	2.5522	6.3937	55.9220
$h=0.090$; $(q_i)_1=1.36$; $(q_i)_2=1.3924$; $(q_i)_3=1.3866$; $(q_i)_{exact}=1.3857$															
First	1.0013	1.0055	1.0128	1.0241	1.0317	1.0409	1.0522	1.0662	1.0836	1.1060	1.1355	1.1765	1.2378	1.3426	1.5830
Second	1.0020	1.0083	1.0198	1.0384	1.0514	1.0679	1.0892	1.1173	1.1552	1.2092	1.2913	1.4288	1.6979	2.3934	5.8044
Third	1.0022	1.0090	1.0217	1.0430	1.0586	1.0792	1.1072	1.1470	1.2068	1.3047	1.4851	1.8810	3.0178	8.2079	75.9740
$h=0.100$; $(q_i)_1=1.40$; $(q_i)_2=1.4400$; $(q_i)_3=1.4320$; $(q_i)_{exact}=1.4307$															
First	1.0014	1.0059	1.0138	1.0260	1.0342	1.0442	1.0564	1.0714	1.0903	1.1144	1.1462	1.1905	1.2567	1.3697	1.6293
Second	1.0022	1.0093	1.0221	1.0430	1.0577	1.0763	1.1004	1.1321	1.1755	1.2374	1.3320	1.4914	1.8054	2.6235	6.6672
Third	1.0024	1.0102	1.0247	1.0491	1.0672	1.0913	1.1244	1.1718	1.2442	1.3643	1.5896	2.0923	3.5591	10.3480	99.8730

TABLE IV.—RATIO OF MINIMUM VELOCITIES FOR COMPRESSIBLE AND INCOMPRESSIBLE FLOWS

Approximation \ M_1	q_{min_c}/q_{min_i}														
	0.10	0.20	0.30	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
$h=0.010$; $(q_i)_1=0.96$; $(q_i)_2=0.9604$; $(q_i)_3=0.9604$; $(q_i)_{exact}=0.9604$															
First	0.9998	0.9991	0.9980	0.9962	0.9950	0.9936	0.9918	0.9896	0.9868	0.9832	0.9787	0.9722	0.9626	0.9401	0.9082
Second	.9998	.9992	.9981	.9965	.9954	.9941	.9925	.9904	.9882	.9852	.9815	.9768	.9709	.9651	.9841
Third	.9998	.9992	.9981	.9965	.9954	.9940	.9924	.9904	.9881	.9850	.9812	.9750	.9683	.9535	.9451
$h=0.015$; $(q_i)_1=0.94$; $(q_i)_2=0.9409$; $(q_i)_3=0.9409$; $(q_i)_{exact}=0.9409$															
First	0.9997	0.9987	0.9969	0.9942	0.9924	0.9901	0.9874	0.9840	0.9798	0.9745	0.9673	0.9575	0.9427	0.9174	0.8694
Second	.9997	.9988	.9972	.9948	.9932	.9913	.9900	.9860	.9829	.9788	.9739	.9680	.9619	.9605	1.0336
Third	.9997	.9988	.9972	.9948	.9932	.9912	.9889	.9858	.9825	.9780	.9726	.9650	.9529	.9214	.5551
$h=0.020$; $(q_i)_1=0.92$; $(q_i)_2=0.9216$; $(q_i)_3=0.9217$; $(q_i)_{exact}=0.9217$															
First	0.9996	0.9982	0.9958	0.9921	0.9896	0.9866	0.9828	0.9783	0.9725	0.9652	0.9555	0.9420	0.9219	0.8875	0.8085
Second	.9996	.9984	.9964	.9932	.9911	.9187	.9857	.9818	.9781	.9731	.9674	.9613	.9508	.9667	------
Third	.9996	.9984	.9963	.9931	.9910	.9885	.9854	.9813	.9772	.9716	.9643	.9538	.9350	.8708	------
$h=0.025$; $(q_i)_1=0.90$; $(q_i)_2=0.9025$; $(q_i)_3=0.9026$; $(q_i)_{exact}=0.9026$															
First	0.9994	0.9977	0.9946	0.9899	0.9867	0.9828	0.9781	0.9722	0.9649	0.9555	0.9431	0.9259	0.9002	0.8562	0.7553
Second	.9995	.9981	.9955	.9917	.9892	.9862	.9827	.9779	.9738	.9682	.9622	.9567	.9559	.9827	------
Third	.9995	.9981	.9955	.9915	.9889	.9858	.9821	.9769	.9721	.9651	.9559	.9418	.9126	.7914	------
$h=0.030$; $(q_i)_1=0.88$; $(q_i)_2=0.8836$; $(q_i)_3=0.8838$; $(q_i)_{exact}=0.8838$															
First	0.9993	0.9972	0.9934	0.9876	0.9837	0.9789	0.9731	0.9659	0.9569	0.9454	0.9302	0.9091	0.8775	0.8235	0.6097
Second	.9994	.9977	.9947	.9902	.9873	.9839	.9799	.9742	.9700	.9642	.9583	.9544	.9595	1.0090	------
Third	.9994	.9977	.9946	.9900	.9869	.9833	.9789	.9725	.9670	.9587	.9472	.9282	.8830	.0721	------
$h=0.035$; $(q_i)_1=0.86$; $(q_i)_2=0.8649$; $(q_i)_3=0.8652$; $(q_i)_{exact}=0.8652$															
First	0.9992	0.9966	0.9921	0.9852	0.9805	0.9748	0.9679	0.9593	0.9486	0.9348	0.9167	0.8915	0.8538	0.7893	0.6415
Second	.9994	.9974	.9940	.9889	.9856	.9818	.9773	.9709	.9668	.9610	.9559	.9545	.9680	1.0483	------
Third	.9994	.9973	.9938	.9885	.9849	.9807	.9757	.9681	.9620	.9521	.9377	.9121	.8439	.5008	------
$h=0.040$; $(q_i)_1=0.84$; $(q_i)_2=0.8464$; $(q_i)_3=0.8469$; $(q_i)_{exact}=0.8469$															
First	0.9990	0.9961	0.9908	0.9827	0.9772	0.9705	0.9624	0.9524	0.9398	0.9238	0.9025	0.8730	0.8289	0.7535	0.5805
Second	.9993	.9971	.9933	.9876	.9840	.9798	.9751	.9679	.9642	.9687	.9549	.9573	.9815	1.0994	------
Third	.9993	.9970	.9930	.9870	.9830	.9783	.9726	.9637	.9569	.9451	.9273	.8926	.7922	.2045	------
$h=0.045$; $(q_i)_1=0.82$; $(q_i)_2=0.8281$; $(q_i)_3=0.8288$; $(q_i)_{exact}=0.8288$															
First	0.9989	0.9955	0.9894	0.9800	0.9737	0.9660	0.9567	0.9451	0.9307	0.9121	0.8876	0.8537	0.8028	0.7159	0.5165
Second	.9992	.9968	.9926	.9864	.9826	.9781	.9731	.9653	.9622	.9574	.9556	.9629	1.0005	------	------
Third	.9992	.9967	.9922	.9855	.9811	.9758	.9695	.9592	.9516	.9377	.9154	.8688	.7251	------	------
$h=0.050$; $(q_i)_1=0.80$; $(q_i)_2=0.8100$; $(q_i)_3=0.8110$; $(q_i)_{exact}=0.8109$															
First	0.9987	0.9948	0.9879	0.9772	0.9701	0.9613	0.9507	0.9375	0.9210	0.8999	0.8720	0.8333	0.7754	0.6765	0.4494
Second	.9991	.9965	.9920	.9854	.9813	.9766	.9715	.9631	.9610	.9573	.9580	.9715	1.0253	------	------
Third	.9991	.9963	.9914	.9841	.9793	.9734	.9664	.9545	.9461	.9296	.9016	.8395	.6392	------	------
$h=0.060$; $(q_i)_1=0.76$; $(q_i)_2=0.7744$; $(q_i)_3=0.7761$; $(q_i)_{exact}=0.7759$															
First	0.9984	0.9935	0.9848	0.9712	0.9622	0.9512	0.9377	0.9211	0.9002	0.8736	0.8384	0.7895	0.7163	0.5913	0.3045
Second	.9990	.9960	.9909	.9836	.9792	.9744	.9693	.9645	.9608	.9604	.9684	.9984	1.0937	------	------
Third	.9989	.9957	.9900	.9813	.9756	.9686	.9601	.9491	.9339	.9105	.8666	.7600	.3964	------	------
$h=0.070$; $(q_i)_1=0.72$; $(q_i)_2=0.7396$; $(q_i)_3=0.7423$; $(q_i)_{exact}=0.7419$															
First	0.9980	0.9920	0.9812	0.9646	0.9534	0.9398	0.9232	0.9028	0.8772	0.8443	0.8010	0.7407	0.6507	0.4967	0.1435
Second	.9989	.9956	.9901	.9824	.9779	.9732	.9687	.9652	.9641	.9689	.9873	1.0397	1.1903	------	------
Third	.9988	.9951	.9885	.9786	.9719	.9637	.9534	.9396	.9195	.8859	.8181	.6438	.0322	------	------
$h=0.080$; $(q_i)_1=0.68$; $(q_i)_2=0.7056$; $(q_i)_3=0.7097$; $(q_i)_{exact}=0.7090$															
First	0.9976	0.9903	0.9773	0.9571	0.9436	0.9272	0.9071	0.8824	0.8513	0.8116	0.7591	0.6863	0.5773	0.3910	0.0305
Second	.9988	.9953	.9895	.9818	.9775	.9734	.9700	.9685	.9713	.9833	1.0158	1.0976	------	------	------
Third	.9986	.9945	.9871	.9758	.9681	.9585	.9459	.9285	.9016	.8537	.7515	.4790	------	------	------
$h=0.090$; $(q_i)_1=0.64$; $(q_i)_2=0.6724$; $(q_i)_3=0.6782$; $(q_i)_{exact}=0.6771$															
First	0.9972	.09884	0.9728	0.9488	0.9326	0.9130	0.8890	0.8594	0.8223	0.7748	0.7121	0.6250	0.4947	0.2720	0.2389
Second	.9988	.9951	.9893	.9819	.9782	.9750	.9733	.9749	.9831	1.0046	1.0552	1.1742	------	------	------
Third	.9985	.9939	.9857	.9730	.9642	.9528	.9375	.9152	.8790	.8113	.6611	.2520	------	------	------
$h=1.000$; $(q_i)_1=0.60$; $(q_i)_2=0.6400$; $(q_i)_3=0.6480$; $(q_i)_{exact}=0.6462$															
First	0.9966	0.9863	0.9678	0.9393	0.9202	0.8969	0.8684	0.8333	0.7894	0.7331	0.6588	0.5555	0.4011	0.1372	0.4084
Second	.9988	.9951	.9895	.9829	.9800	.9783	.9791	.9848	1.0000	1.0338	1.1071	--- --	------	------	------
Third	.9984	.9933	.9843	.9700	.9599	.9464	.9276	.8990	.8504	.7559	.5407	------	------	------	------

TABLE V.—VALUES OF FLUID VELOCITY AS A FUNCTION OF STREAM MACH NUMBER FOR VARIOUS CONSTANT VALUES OF LOCAL MACH NUMBER

	q														
M_1 \ M	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00 (q_{cr})	1.05	1.10	1.15	1.20	∞ (vacuum)
0.10	1.00000	1.99398	2.97635	3.94145	4.88443	5.80079	6.68702	7.53997	8.35744	9.13783	9.51380	9.88020	10.23699	10.58418	22.38303
.20	.50150	1.00000	1.49263	1.97662	2.44951	2.90907	3.35351	3.78125	4.19121	4.58258	4.77112	4.95487	5.13380	5.30791	11.22497
.30	.33598	.66994	1.00000	1.32426	1.64109	1.94897	2.24673	2.53331	2.80797	3.07017	3.19649	3.31959	3.43947	3.55612	7.52034
.40	.25371	.50591	.75514	1.00000	1.23925	1.47175	1.69660	1.91300	2.12041	2.31840	2.41379	2.50675	2.59728	2.68536	5.67891
.50	.20474	.40824	.60936	.80695	1.00000	1.18762	1.36906	1.54369	1.71106	1.87083	1.94780	2.02282	2.09586	2.16694	4.58258
.60	.17239	.34374	.51310	.67947	.84203	1.00000	1.15277	1.29982	1.44074	1.57527	1.64009	1.70325	1.76475	1.82461	3.85861
.70	.14953	.29819	.44510	.58942	.73044	.86747	1.00000	1.12756	1.24981	1.36651	1.42273	1.47753	1.53088	1.58280	3.34725
.80	.13263	.26446	.39474	.52274	.64780	.76934	.88688	1.00000	1.10842	1.21192	1.26178	1.31038	1.35769	1.40374	2.96859
.90	.11967	.23858	.35613	.47161	.58444	.69409	.80013	.90219	1.00000	1.09338	1.13836	1.18220	1.22490	1.26644	2.67822
1.00	.10945	.21822	.32571	.43134	.53453	.63481	.73179	.82514	.91460	1.00000	1.04114	1.08124	1.12028	1.15828	2.44949
1.05	.10512	.20959	.31284	.41428	.51340	.60972	.70787	.79253	.87845	.96048	1.00000	1.03851	1.07602	1.11251	2.35269
1.10	.10119	.20182	.30125	.39892	.49437	.58711	.67681	.76314	.84588	.92486	.96292	1.00000	1.03611	1.07125	2.26545
1.15	.09767	.19478	.29074	.38502	.47714	.56665	.65322	.73654	.81639	.89262	.92935	.96515	1.00000	1.03391	2.18600
1.20	.09450	.18839	.28121	.37238	.46149	.54806	.63179	.71238	.78961	.86335	.89887	.93348	.96720	1.00000	2.11476

TABLE VI.—VALUES OF CRITICAL STREAM MACH NUMBER FOR VARIOUS VALUES OF CAMBER COEFFICIENT

k	$M_{1_{cr}}$		
	Approximation		
	First	Second	Third
0.02	0.848	0.832	0.825
.04	.770	.746	.738
.06	.716	.682	.672
.08	.670	.628	.620
.10	.625	.585	.574

TABLE VII.—VALUES OF MAXIMUM VELOCITY FOR CORRESPONDING BUMP AND CIRCULAR ARC PROFILE

M	q_{max}									
	Camber coefficient, k					Thickness coefficient, t				
	0.02	0.04	0.06	0.08	0.10	0.052	0.100	0.145	0.186	0.226
0	1.0815	1.1659	1.2527	1.3415	1.4320	1.0816	1.1660	1.2527	1.3414	1.4320
.2	1.0834	1.1701	1.2597	1.3520	1.4466	1.0834	1.1701	1.2595	1.3513	1.4454
.3	1.0859	1.1759	1.2695	1.3668	1.4673	1.0859	1.1757	1.2689	1.3651	1.4641
.4	1.0899	1.1851	1.2855	1.3913	1.5024	1.0900	1.1847	1.2840	1.3876	1.4950
.5	1.0960	1.1997	1.3116	1.4324	1.5627	1.0959	1.1988	1.3084	1.4245	1.5467
.6	1.1056	1.2239	1.3572	1.5078	1.6780	1.1052	1.2217	1.3492	1.4879	1.6373
.7	1.1223	1.2705	1.4530	1.6780	--------	1.1213	1.2640	1.4298	1.6197	--------
.8	1.1594	1.3979	--------	--------	--------	1.1557	1.3701	--------	--------	--------
.9	1.2055	--------	--------	--------	--------	1.1960	--------	--------	--------	--------

TABLE VIII.—VALUES OF THE COEFFICIENTS a_1, a_2, a_3, a_4, AND a_5 OBTAINED FROM EQUATION (25)

M_1	β	D	$G_1(0)$	$G_2(0)$	a_1	a_2	a_3	a_4	a_5
0	1.00000	0	8.00000	−4.00000	4.00000	0	−4.00000	−8.00000	0
.1	.99499	.01010	8.09648	−4.03077	4.02016	.04064	−4.08129	−7.90776	−.24568
.2	.97980	.04167	8.42740	−4.13336	4.08248	.17085	−4.34170	−7.52224	−1.05994
.3	.95394	.09890	9.14836	−4.34613	4.19312	.41907	−4.83814	−6.56184	−2.73342
.4	.91652	.19048	10.67678	−4.77006	4.36432	.84900	−5.69800	−4.18200	−6.03824
.45	.89303	.25392	12.04762	−5.13375	4.47912	1.17042	−6.34085	−1.83776	−8.86482
.50	.86603	.33333	14.17991	−5.68239	4.61876	1.60000	−7.20000	2.02300	−13.12718
.55	.83516	.43369	17.63525	−6.54590	4.78952	2.18617	−8.37234	8.59476	−19.84720
.60	.80000	.56250	23.53305	−7.97811	5.00000	3.00938	−10.01875	20.30732	−31.02488
.65	.75993	.73160	34.27894	−10.51506	5.26364	4.21097	−12.42194	42.46800	−50.90758
.70	.71414	.96078	55.59189	−15.40352	5.60116	6.05856	−16.11711	87.93708	−89.39792
.75	.66144	1.28571	102.96868	−25.93794	6.04740	9.11016	−22.22032	192.27640	−172.73056
.80	.60000	1.77778	226.68292	−52.51419	6.66667	14.69630	−33.39260	473.2848	−383.84688
.85	.52678	2.60360	640.44954	−137.91532	7.59332	26.68336	−57.36672	1443.289	−1064.5217
.90	.43589	4.26316	2779.25502	−558.79732	9.17664	60.67151	−125.34302	6628.288	−4426.698
.95	.31225	9.25641	33380.52469	−6230.9282	12.81024	242.66034	−489.32067	83649.05	−49791.19

TABLE IX.—VELOCITY DISTRIBUTION AT UPPER AND LOWER SURFACES OF CIRCULAR ARC PROFILE, $h=0.05$

	q							
	Upper surface				Lower surface			
x \ M_1	0	0.3	0.5	0.7	0	0.3	0.5	0.7
1	0.9900	0.9890	0.9860	0.9749	0.9900	0.9890	0.9860	0.9749
.95	1.0541	1.0566	1.0604	1.0638	.9298	.9260	.9187	.9016
.90	1.0805	1.0842	1.0920	1.1061	.9071	.9029	.8937	.8742
.80	1.1166	1.1226	1.1361	1.1680	.8778	.8727	.8618	.8397
.70	1.1423	1.1502	1.1683	1.2161	.8581	.8524	.8404	.8158
.60	1.1620	1.1714	1.1934	1.2553	.8436	.8375	.8247	.7975
.50	1.1773	1.1880	1.2132	1.2874	.8327	.8262	.8128	.7832
.40	1.1892	1.2008	1.2287	1.3130	.8244	.8178	.8038	.7721
.30	1.1980	1.2104	1.2403	1.3327	.8184	.8115	.7972	.7637
.20	1.2042	1.2171	1.2484	1.3466	.8142	.8073	.7927	.7579
.10	1.2078	1.2210	1.2532	1.3549	.8118	.8049	.7900	.7544
0	1.2090	1.2223	1.2548	1.3577	.8110	.8040	.7892	.7532

TABLE X.—CONVERSION FROM FLUID VELOCITY q TO PRESSURE COEFFICIENT C_{p, M_1}, FOR VARIOUS VALUES OF STREAM MACH NUMBER M_1

	C_{p, M_1}							
q \ M_1	0	0.1	0.2	0.3	0.4	0.45	0.50	0.55
0.25	0.9375	0.93971	0.94632	0.95743	0.97319	0.98285	0.99373	1.00587
.30	.9100	.91200	.91836	.92879	.94361	.95270	.96294	.97436
.35	.8775	.87943	.88525	.89495	.90873	.91719	.92669	.93729
.40	.8400	.84186	.84707	.85600	.86861	.87633	.88503	.89473
.45	.7975	.79914	.80366	.81190	.82327	.83022	.83805	.84677
.50	.7500	.75143	.75564	.76275	.77278	.77891	.78582	.79351
.55	.6975	.69871	.70239	.70851	.71718	.72248	.72844	.73508
.60	.64	.64100	.64407	.64927	.65655	.66100	.66601	.67157
.65	.5775	.57829	.58086	.58503	.59096	.59459	.59865	.60317
.70	.5100	.51071	.51261	.51587	.52049	.52330	.52647	.52997
.75	.4375	.43800	.43943	.44183	.44521	.44728	.44959	.45217
.80	.3600	.36029	.36129	.36292	.36521	.36679	.36817	.36991
.85	.2775	.27771	.27825	.27924	.28060	.28143	.28234	.28338
.90	.1900	.19014	.19036	.19081	.19145	.19184	.19227	.19275
.95	.0975	.09757	.09761	.09771	.09788	.09799	.09809	.09822
1.00	0	0	0	0	0	0	0	0
1.05	−.1025	−.10243	−.10239	−.10225	−.10208	−.10197	−.10185	−.10170
1.10	−.2100	−.20986	−.20954	−.20900	−.20824	−.20777	−.20726	−.20669
1.15	−.3225	−.32229	−.32146	−.32016	−.31836	−.31726	−.31605	−.31471
1.20	−.4400	−.43943	−.43807	−.43565	−.43230	−.43029	−.42803	−.42555
1.25	−.5625	−.56171	−.55932	−.55541	−.54996	−.54666	−.54300	−.53898
1.30	−.6900	−.68871	−.68525	−.67935	−.67117	−.66623	−.66075	−.65474
1.35	−.8225	−.82086	−.81575	−.80740	−.79579	−.78882	−.78109	−.77260
1.40	−.9600	−.95757	−.95082	−.93944	−.92370	−.91425	−.90378	−.89231
1.45	−1.1025	−1.09943	−1.09039	−1.07543	−1.05473	−1.04233	−1.02861	−1.01362
1.50	−1.2500	−1.24614	−1.23446	−1.21524	−1.18875	−1.17289	−1.15537	−1.13626
1.55	−1.4025	−1.39757	−1.38293	−1.35879	−1.32558	−1.30573	−1.28383	−1.25994
1.60	−1.5600	−1.55386	−1.53582	−1.50602	−1.46507	−1.44066	−1.41377	−1.38454
1.65	−1.7225	−1.71500	−1.69304	−1.65678	−1.60706	−1.57748	−1.54496	−1.50965
1.70	−1.8900	−1.88100	−1.85454	−1.81098	−1.75140	−1.71601	−1.67717	−1.63508
1.75	−2.0625	−2.05186	−2.02029	−1.96856	−1.89791	−1.85604	−1.81016	−1.76067

	C_{p, M_1}							
q \ M_1	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
0.25	1.01929	1.03405	1.05017	1.06770	1.08669	1.10719	1.12925	1.15203
.30	.98699	1.00086	1.01601	1.03249	1.05032	1.06957	1.09027	1.11250
.35	.94901	.96188	.97592	.99119	1.00772	1.02554	1.04464	1.06520
.40	.90544	.91720	.93003	.94397	.95904	.97530	.99277	1.01150
.45	.85640	.86697	.87848	.89100	.90452	.91909	.93474	.95151
.50	.80200	.81131	.82146	.83247	.84437	.85718	.87093	.88565
.55	.74239	.75041	.75915	.76863	.77885	.78986	.80166	.81429
.60	.67772	.68444	.69176	.69969	.70825	.71744	.72730	.73783
.65	.60814	.61360	.61952	.62594	.63285	.64028	.64823	.65672
.70	.53384	.53807	.54266	.54763	.55298	.55873	.56487	.57142
.75	.45500	.45810	.46145	.46509	.46899	.47318	.47765	.48241
.80	.37181	.37390	.37616	.37860	.38122	.38402	.38701	.39020
.85	.28450	.28573	.28706	.28850	.29004	.29169	.29345	.29531
.90	.19327	.19384	.19446	.19513	.19585	.19661	.19742	.19829
.95	.09836	.09851	.09867	.09885	.09903	.09923	.09944	.09967
1.00	0	0	0	0	0	0	0	0
1.05	−.10156	−.10139	−.10122	−.10103	−.10083	−.10061	−.10039	−.10015
1.10	−.20606	−.20538	−.20465	−.20387	−.20304	−.20216	−.20122	−.20021
1.15	−.31325	−.31166	−.30996	−.30813	−.30620	−.30415	−.30198	−.29971
1.20	−.42285	−.41993	−.41679	−.41345	−.40989	−.40613	−.40218	−.39804
1.25	−.53460	−.52987	−.52480	−.51940	−.51368	−.50705	−.50131	−.49169
1.30	−.64821	−.64117	−.63363	−.62562	−.61715	−.60824	−.59890	−.58916
1.35	−.76340	−.75350	−.74293	−.73171	−.71983	−.70746	−.69448	−.68098
1.40	−.87990	−.86656	−.85235	−.83730	−.82146	−.80488	−.78760	−.76968
1.45	−.99740	−.98002	−.96153	−.94200	−.92150	−.90008	−.87784	−.85185
1.50	−1.11563	−1.09356	−1.07014	−1.04545	−1.01959	−.99268	−.96480	−.93608
1.55	−1.23429	−1.20686	−1.17781	−1.14727	−1.11538	−1.08229	−1.04811	−1.01301
1.60	−1.35310	−1.31961	−1.28423	−1.24713	−1.20851	−1.16856	−1.12745	−1.08542
1.65	−1.47177	−1.43149	−1.38909	−1.34469	−1.29861	−1.25116	−1.20250	−1.15295
1.70	−1.59001	−1.54221	−1.49199	−1.43962	−1.38545	−1.32980	−1.27301	−1.21544
1.75	−1.70754	−1.65147	−1.59270	−1.53163	−1.46867	−1.40423	−1.33876	−1.27271